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A Duality Theory for the Algebraic Invariants of Substitution Tiling Spaces

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by

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A duality theory for the algebraic invariants of substitution tiling spaces

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We present here a method for computing the homology of a substitution tiling space. There is a well established cohomology theory that uses simple matrix computations to determine if two tiling spaces are different. We will show how to compute Putnam's homology groups for these spaces using simple linear algebra. We construct a Markov Partition based on the substitution rules, and exploit the properties of this partition as a shift of finite type to construct algebraic invariants for the tiling space. These invariants form a chain complex, of which we can compute the homology. In our examples we will demonstrate an interesting duality between the cohomology and homology of these spaces. This leads to a conjecture relating the two theories to each other and we present the reasoning behind the conjecture.

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Chapter 1

Introduction

Substitution tilings are interesting mathematical objects, both from a purely dynamical perspective, and from their relation to quasicrystals, materials science, and computer science. We are interested in classifying these tilings by knowing when they are combinatorially equivalent. By building a topological space of these tilings, called a tiling space, we are able to apply topological invariants to the spaces in question. If we are able to find that two spaces are topologically homeomorphic via an orientation preserving map, we know then that the tilings are combinatorially equivalent. The cohomology of these tiling spaces has been well understood for some time [1]. Recently a homology theory has been presented which can be applied to these spaces. Here we will consider one-dimensional substitution tiling spaces, with finite local complexity. We employ the homology theory of Putnam [13] for Smale spaces to compute the homology of our tiling spaces.

To understand tiling spaces, it is appropriate to begin with Smale's Axiom A systems [18]. An Axiom A system is a map f on a smooth manifold M, satisfying the conditions that

- The non-wandering set of f, $\Omega(f)$ is hyperbolic and compact.

- The periodic points of f are dense in $\Omega(f)$.

These were the foundation for Ruelle's [16] definition of a Smale Space. Ruelle added a binary operation called the bracket that can be applied to two points in an Axiom A system that are sufficiently close using the appropriate metric. Ruelle defined stable and unstable sets for each point in the space, and showed how the bracket operation was the unique point of intersection of the stable and unstable sets of the points in question. It was shown later by Bowen [6] that the non-wandering sets of the Axiom A systems admitted a Markov partition. This allowed the systems to be expressed as a coding known as a shift of finite type [11] when the automorphism on the system is restricted to certain subsets. It is viewing a tiling space in this manner that allows the use of Putnam's [13] homology theory.

Smale also described the dynamics of the solenoid [18] under the doubling map. This map essentially begins with a torus, then embeds this torus inside itself by wrapping n times. The map is then repeated. This was adapted by Williams' [19] in his description of one-dimensional hyperbolic attractors, essentially using copies of the circle instead of the torus and constructing the solenoid as an inverse limit. There is a relationship between these solenoids and aperiodic tiling spaces. The solenoids are inverse limits of expanding maps on circles, while the tiling spaces are inverse limits of expanding maps on wedges of circles.

All hyperbolic one-dimensional attractors can be split into two classes. There are the true solenoids [19], which are purely distinguishable by their cohomology. There are also the aperiodic substitution tiling spaces [1, 4, 20], which have been referred to as degenerate solenoids or as Williams' solenoids. Anderson and Putnam [1] showed that the unstable equivalence classes of a Smale space are actually the orbits of a tiling dynamical system. From a topological perspective, these are really the arc components of the space. In this way, the properties of the dynamical system were able to be used to compute topological invariants. Anderson and Putnam [1], and later Barge and Diamond [3], used this to view a tiling space as an inverse limit space, which allowed them to compute the Čech cohomology of the spaces.

One-dimensional substitution tiling spaces are constructed by taking a collection of sets (called prototiles) which are homeomorphic to the unit ball on the real line, and applying to them a substitution rule which inflates each tile and replaces the inflated tile with a collection of prototiles. The tiling space is the collection of all possible allowed tilings of the real line using these particular prototiles. By an allowed tiling, we mean a patch of prototiles that may arise from action on a set of prototiles by substitution. There are two maps on the tiling space, the substitution map and the map which acts on a tiling by translation, essentially moving the origin. Both of these maps are automorphisms of the tiling space.

The method described here for computing the homology of these spaces involves constructing a Markov partition which maps to the tiling space via a finite-to-one, onto, map. We will record the location of the origin at each iteration of the substitution, and in doing so, create a sequence that can be mapped to the tiling. The map is not one-to-one, since at the boundary of two tiles, we have two possible choices of coding. Once we have made a choice (right or left), we make the same choice for all future codings. Depending on the tiling we are using, it may not be sufficient to only know in which prototile the origin has landed, but also the j nearest neighbors of this prototile. Knowing these neighbors makes our tiling recognizable, in that if we know what tile we are in, we know where we came from prior to the last substitution.

Once we have our Markov partition constructed, we can use properties of shifts of finite type [11] to define algebraic invariants known as dimension groups [9]. It can be shown that these groups form a chain complex, of which we may take the homology.

It is conjectured that, for all one-dimensional substitution tiling spaces Ω, the homology and Čech cohomology are related via

$$H_0(\Omega) \cong \check{H}^1(\Omega) \text{ and } H_1(\Omega) \cong \check{H}^0(\Omega).$$

This leads us to suspect a higher analog for n-dimensional aperiodic substitution tiling spaces.

Conjecture 1.1. *Given an aperiodic substitution tiling space Ω, of dimension n,*

$$H_k(\Omega) \cong \check{H}^{n-k}(\Omega).$$

We present several examples which support this theory for one-dimensional spaces, and show the future direction of our work in confirming it.

In this paper, we first present aperiodic substitution tiling spaces, showing that they are both Smale spaces and inverse limit spaces. As inverse limit spaces we present a method for computing cohomology, and as Smale spaces, we present a method for computing homology. Much of our work here is in simplifying the theory of [13] to apply specifically to substitution tiling spaces. Finally, we produce examples in which we support the conjecture above.

Chapter 2

Substitution tiling spaces

We consider here the dynamical systems generated by substitution tiling systems. A tiling of a space $\mathbb{R}^d$ is a covering of the space by sets with pairwise disjoint interiors, each of which is a translation of one of a finite number of sets called *prototiles* [1]. Each prototile is a set which is homeomorphic to an open ball in $\mathbb{R}^d$. We begin by considering *partial tilings*, which are collections of tile that have pairwise disjoint interiors. The *support* of a partial tiling is the union of it's tiles. Thus a tiling T has support $\mathbb{R}^d$. For a tile t, we use the notation $t \in T$.

Definition 2.1. *[17] A tiling is* ***rotationally simple*** *if it satisfies three assumptions*

1. *There are only a finite number of prototiles, up to Euclidean motion.*
2. *Each tile is a polygon*
3. *Tiles meet full-edge to full-edge*

For our purposes we will only consider tilings which are rotationally simple.

Let $u \in \mathbb{R}^d$ and $U \subseteq \mathbb{R}^d$. Then we have

$$T(u) = \{t \in T | u \in t\}. \quad T(U) = \bigcup_{u \in U} T(u).$$

We define expansions and translations of T by

$$\lambda T = \{\lambda t | t \in T\} \text{ for } \lambda \in \mathbb{R}^+$$

$$T + u = \{t + u | t \in T\} \text{ for } u \in \mathbb{R}^d.$$

Often a collection of tilings Ω is defined by a substitution rule. Let $\{p_i | i = 1, \dots, n\}$ be the set of *prototiles*. Let $\hat{\Omega}$ be the collection of all partial tilings of $\mathbb{R}^d$ that contain only translations of these prototiles. We assume that there is some inflation constant $\lambda > 1$, and a rule that associates each prototile with a partial tiling, such that when it is inflated by λ, it is in $\hat{\Omega}$. We define the inflation map $\hat{\omega} : \hat{\Omega} \to \hat{\Omega}$ by

$$\hat{\omega}(T) = \lambda \bigcup_{p_i + u \in T} (P_i + u).$$

We let Ω be the subset of $\hat{\Omega}$ such that for any partial tiling P with bounded support, we have $P \subseteq \hat{\omega}^n(\{p_i + u\})$. Thus P is a subset of it's component prototiles after they have been shifted and inflated appropriately. We can then let $\omega = \hat{\omega}|_{\Omega}$. This map takes a tiling and applies the inflation map and substitution rule to it, but the result is another tiling in Ω.

As an example, we may look at the Fibonacci tiling of $\mathbb{R}$. Think of two sets homeomorphic to open balls that are placed on the real line, and labelled a and b. These are really just intervals in this case. We may assume that they are of different lengths. We define the substitution rule as

$$a \to ab$$

$$b \to a.$$

In such a way we may inflate each tile and substitute the appropriate prototiles to obtain another tiling of the real line. Only certain sequences of prototiles may occur in a tiling, depending on the substitution rule. We say a patch of tiles is *allowed* if it occurs as the result of applying the substitution map to the prototiles.

The space Ω in this case is the collection of all allowed tilings of the line, and the map $\omega : \Omega \to \Omega$ can be thought of as simply shifting the location of the origin to obtain another element of the space.

Our conjecture will rely on a theory for a more general type of dynamical system, called a *Smale space*. A Smale space is a non-empty, compact, metric space Ω, with a map $\omega : \Omega \to \Omega$. We generally denote such a space as simply (Ω, ω). Assuming we are given an $\epsilon > 0$, we define a map $[\cdot, \cdot]$ called the *bracket* as

$$[\cdot, \cdot] : \{(x, y) \in \Omega \times \Omega | d(x, y) < \epsilon\} \to \Omega$$

with the properties that the bracket is continuous and

- $[x, x] = x$
- $[[x, y], z] = [x, z]$
- $[x, [y, z]] = [x, z]$

when all operations above are defined. A Smale space has local stable and unstable sets defined by

$$V^S(x, \epsilon) = \{u | u = [u, x] \text{ and } d(x, u) < \epsilon\}$$

$$V^U(x, \epsilon) = \{v | v = [x, v] \text{ and } d(x, v) < \epsilon\}$$

with the property that

$$[x, y] = V^S(x, \epsilon) \cap V^U(x, \epsilon).$$

We will also require the global stable and unstable sets, defined by

$$V^S(x) = \bigcup_{n=0}^{\infty} \omega^{-n}(V^S(\omega^n(x), \epsilon))$$

$$V^U(x) = \bigcup_{n=0}^{\infty} \omega^n(V^S(\omega^{-n}(x), \epsilon))$$

For the formal definition, see [16].

In order to show that our tiling space is a Smale space, we must first define a metric on the tiling space. Given any two tilings T, $T' \in \Omega$, we define

$$d(T,T') = \inf(\{1/\sqrt{2}\} \cup \{\epsilon | T + u \text{ and } T' + v \text{ agree on } B_{1/\epsilon}(0) \text{ for some } \|u\|, \|v\| < \epsilon\}).$$

Essentially, two tilings are close if they agree on a ball of radius $1/\epsilon$ around the origin after a translation of at most ϵ [1].

With our metric defined, we let (Ω, d, ω) be our tiling space. We define the bracket for our tiling space as $[T, T'] = T' + v - u$, when $d(T, T') < \epsilon_0$.

By [1] we have that (Ω, d, ω) is a Smale space. We note here that Ω is the closure of $\hat{\Omega}$, and that the space contains only one connected component. Putnam [13] has developed a homology theory that is valid for any Smale space. We will show how the general theory may be simplified for a tiling space. Substitution tiling spaces have the property of being non-wandering, which will aid us in the simplification.

Definition 2.2. *A point x in a dynamical system (Ω, ω) is* ***non-wandering*** *if, given an open set U containing x, there is a positive integer N such that $\omega^N(U) \cap U$ is non-empty. We say the system (Ω, ω) is non-wandering if each $x \in \Omega$ is non-wandering.*

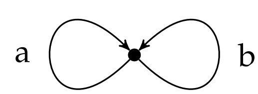
Another property of tiling spaces we will be using in computing both the cohomology and the homology is that tiling spaces can be viewed as inverse limit spaces, which we now define.

Begin with a collection $\Gamma_0, \Gamma_1, \ldots$ of topological spaces. For each $n \in \mathbb{Z}_{\geq 0}$, let $f_n : \Gamma_{n+1} \to \Gamma_n$ be a continuous map. We view the product space $\Pi\Gamma_n$ as a set of sequences $(x_0, x_1, \ldots)$ with each $x_n \in \Gamma_n$.

Definition 2.3. *[17] The* ***inverse limit space*** *of a collection of topological spaces as above is*

$$\varprojlim(\Gamma, f) = \{(x_0, x_1, \ldots) \in \Pi\Gamma_n | \text{ for all } n, x_n = f_n(x_{n+1}\}.$$

As per [1], for any tiling space, we may consider it as an inverse limit of CW complexes, using the substitution as the bonding map. For a tiling space the complex Γ_i is fixed, and the map f_i is the substitution map for all i. These complexes can be generated using the prototiles as edges, with the connected components representing when it is possible for one prototile to follow another. For example, in the Fibonacci tiling above, we could have the complex

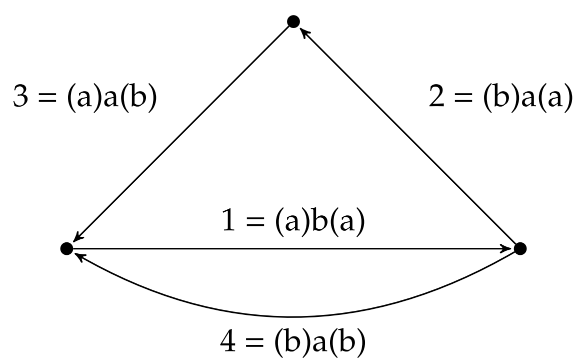


as each of our Γ_n. The Fibonacci tiling space is the inverse limit space taken from these CW-complexes using the bonding map introduced by the substitution rule. That is, the circle a will wrap around itself and then the b circle, and the circle b will wrap around the a circle under the map. The CW complex is a bit misleading, as it appears the path bb can occur here, when it does not occur in the tiling space. Due to this artifact, it may not be the case the the inverse limit space is homeomorphic to the substitution tiling space, but the following theorem gives a condition on the tiling space that will guarantee this property.

Theorem 2.1. *[1] If a substitution forces it's border, then the inverse limit of the component spaces under the substitution map is homeomorphic to the tiling space.*

Definition 2.4. *[8] A substitution tiling space* (Ω, ω) ***forces it's border*** *if, given two tilings* T, T' *and a point* $t \in T$, $t \in T'$*, there exists a positive integer* N *such that* $\omega^N(T)$ *and* $\omega^N(T')$ *coincide. That is, the tiles must have the same pattern of neighboring tiles following the substitution.*

We can always create an equivalent tiling spaces that forces it's border by collaring. Rather than just using the individual prototiles as our edges, we can created *collared* tiles, where we relabel the tiles so that we know not only the tile type, but also that of it's neighbors. For example, rather than having the tiles a and b, let us instead use the tiles $1 = (a)b(a)$, $2 = (b)a(a)$, $3 = (a)a(b)$, and $4 = (b)a(b)$. The letters in parentheses are not part of the tile, but they represent the nearest neighbors to each prototile, and so are included in our label of the tile. Our CW complex then becomes



The inverse limits of these CW complexes are isomorphic [1]. Thus tiling spaces may be viewed as inverse limits of expanding maps on wedges of circles. Such spaces correspond to Williams' solenoids [19].

Chapter 3

Čech cohomology of substitution tiling spaces

There are several methods for computing the Čech cohomology of a tiling space. The first method was given by Anderson and Putnam in [1]. It relies on showing that a tiling space is an inverse limit space, and that the Čech cohomology of an inverse limit space is isomorphic to the direct limit of the singular cohomology of the individual spaces in the inverse limit. That is,

$$\check{H}^n(\varprojlim(\Gamma, \varphi) \cong \varinjlim(H^n(\Gamma), \varphi^*)$$

[12] where φ is the bonding map and φ^* is the inducted map on the cohomology groups of Γ.

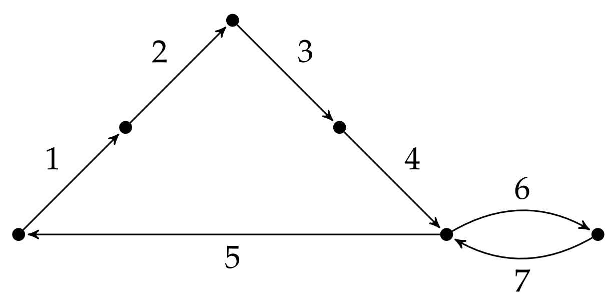
This method may be computationally intensive, as we see when we examining the tiling space (X, φ) generated by

$$a \to aaabb \text{ and } b \to ab.$$

Using the Anderson-Putnam method, we first collar the tiles and relabel them. This gives us 7 tiles, which we list below, as well as showing where they are taken by the substitution map φ.

$$\begin{array}{rcl}
1: & (a)a(a) \xrightarrow{\varphi} (aaabb)aaabb(aaabb) & = 51234 \\
2: & (a)a(b) \xrightarrow{\varphi} (aaabb)aaabb(ab) & = 51234 \\
3: & (a)b(b) \xrightarrow{\varphi} (aaabb)ab(ab) & = 67 \\
4: & (b)b(a) \xrightarrow{\varphi} (ab)ab(aaabb) & = 67 \\
5: & (b)a(a) \xrightarrow{\varphi} (ab)aaabb(aaabb) & = 51234 \\
6: & (b)a(b) \xrightarrow{\varphi} (ab)aaabb(ab) & = 51234 \\
7: & (a)b(a) \xrightarrow{\varphi} (aaabb)ab(aaabb) & = 67
\end{array}$$

Since the tiling space is an inverse limit space, we can take the one CW complex that makes up the inverse limit space and represent it with a graph, as below.



We then look at where each edge of the graph and each vertex of the graph is taken under the substitution. We will use a theorem of [1] that the Čech cohomology of an inverse limit space is isomorphic to the direct limit of the cohomology as stated above. Begin by computing the substitution matrices that records how φ acts on edges and vertices. We will denote these as A_0 and A_1, referring to the vertices and the edges. We must also build a matrix δ_0 which computes the coboundary map of the CW complex.

There are 6 vertices and each vertex is mapped to another vertex under substitution, thus

$$A_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

There are 7 edges and each edge is mapped exactly over a collection of edges. We place a 1 in the ij entry of the matrix if the edges i maps over the edge j. Thus

$$A_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

and since the coboundary of each vertex tracks which edges enter or leave that vertex we have the 7×6 matrix

$$\delta_0 = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 01 & -1 & \end{bmatrix}.$$

We build the following commutative diagram which shows how the coboundary map is related to the matrices which are used to compute the direct limit. We let $C_0 \cong \mathbb{Z}^6$ be the zero dimensional cochains and $C_1 \cong \mathbb{Z}^7$ be the one-dimensional cochains. All higher dimensions are zero in this case.

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\ C^1 \arrow[u, "\delta_1"'] \arrow[r, "A_1"] & C^1 \arrow[u, "\delta_1"'] \arrow[r, "A_1"] & C^1 \arrow[u, "\delta_1"'] \arrow[r, "A_1"] & \dots \\ C^0 \arrow[u, "\delta_0"'] \arrow[r, "A_0"] & C^0 \arrow[u, "\delta_0"'] \arrow[r, "A_0"] & C^0 \arrow[u, "\delta_0"'] \arrow[r, "A_0"] & \dots \end{tikzcd}$$

By computing the eigenvalues and rank of each matrix, it becomes easy to compute the image and kernel of the coboundary map, and thus compute the cohomology.

The ranks of the matrices are $\text{Rank}(A_1) = 2$, $\text{Rank}(A_0) = 2$, and $\text{Rank}(\delta_0) = 5$, and the corresponding eigenvalues are $\lambda_{A_1} = 0, 0, 0, 0, 0, 2 + \sqrt{3}, 2 - \sqrt{3}$ and $\lambda_{A_0} = 0, 0, 0, 0, 0, 1$.

We use the computational method presented in Sadun [17] here. Let K represent the complex above. We compute $H^0(K) \cong \ker(\delta_0)$ and $H^1(K) \cong \ker(\delta_1)/\mathrm{im}(\delta_0)$. To determine what is in each image and kernel, we rely on the fact that the diagram must commute. Therefore, if an eigenspace associated with A_0 is not an eigenspace associated to A_1, it must be in the kernel of δ_0. The range of δ_1 is zero, so everything must be in the kernel. We note that no non-zero eigenvalues will persist in the direct limit, so we need only take the direct limit of the cohomology groups under the matrices A_0 and A_1 respectively. These are easily computed using the eigenvalues, giving us

$$\check{H}^0 \cong \mathbb{Z}$$

$$\check{H}^1 \cong \mathbb{Z}^2.$$

Since there is only one connected component, in all one-dimensional tiling spaces, $\check{H}^0(\Omega) \cong \mathbb{Z}$ as described in Munkres [12]. We see that this is identical to the result obtained above.

While these computations were not difficult in this example, direct limits can be much more complicated. We will instead consider the method developed by Barge and Diamond [3]. It is less computationally intensive, and will be more useful here, since we are only interested in comparing the results of the cohomology computations with the homology computations.

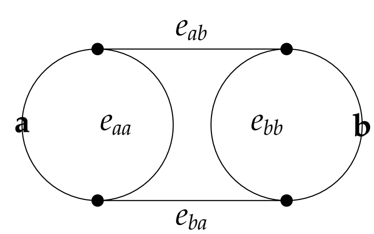
The Barge-Diamond method involves looking at something called the *germ* of each tile. This can be thought of as a small piece of the edge of each tile. By looking at the eventual range of each germ under the substitution, we are able to determine the cohomology of the entire space.

Definition 3.1. *[17] Let W be a finite set, and $f : W \to W$. There exists an $N \in \mathbb{N}$ such* *that for all $n \geq N$, $f^n(W) = f^N(W)$. $f^N(W)$ is called the **eventual range** of W.*

We now take a CW complex X that represents our tiling space Ω, and a map σ representing the the range of each prototile during an iteration of the substitution. Let M be a matrix representing σ. Let X_0 be the set of germs of X, and $(X_0)_{ER}$ it's eventual range under σ. We then have

Theorem 3.1. *[3] For a one-dimensional tiling space, if $(X_0)_{ER}$ has k connected components and l independent loops, then $\check{H}^1(\Omega) \cong \mathbb{Z}^l \oplus \varinjlim A/\mathbb{Z}^{k-1}$.*

Using this method we can compute the cohomology of any one dimensional tiling space quite quickly and easily. Using the example given above, but without collaring, $a \to aaabb$ and $b \to ab$. We will see how this is a much faster and simpler method. We note that aa,ab,ba, and bb are all possible, so there are transition germs e_{aa}, e_{ab}, e_{ba}, and e_{bb}. These can be thought of as small pieces of the prototiles that have been "split" off according to their transitions. In contrast, there would not be an e_{bb} germ in the Fibonacci tiling, as bb does not occur (see Chapter 6). For our current example, the Barge-Diamond complex is



We can apply the substitution rule to see where each germ ends up. The germ e_{aa} represents the "end" of and a tile and the "beginning" of an a tile. The transition aa maps to the transition ba. We therefore say $e_{aa} \to e_{ba}$, and similarly $e_{ab} \to e_{ba}$,

$e_{ba} \to e_{ba}$, and $e_{bb} \to e_{ba}$, since all tiles end in a and begin in b under substitution. Only one germ is in the eventual range, so we have $l = k = 1$. We take the direct limit of the matrix

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

Computing the eigenvalues to be $\lambda = 2 \pm \sqrt{3}$, we have that the direct limit is $\mathbb{Z}^2$. Since the eventual range is only one element, we have one connected component and no loops. Thus the cohomology computation becomes

$$\check{H}^1 \cong \mathbb{Z}^0 \oplus \mathbb{Z}^2/\mathbb{Z}^0 \cong \mathbb{Z}^2.$$

The eventual range may have many loops and connected components. This would prevent the cohomology from being computable merely by the direct limit. We present an example in chapter 6 where this is the case.

Chapter 4

Factor maps, shifts of finite type, and Markov partitions

Here we describe several characteristics of a Smale space that are necessary for the development of the homology theory. All of the following are standard definitions from dynamical systems [11, 15, 7], but we simplify many definitions to apply specifically to substitution tiling spaces.

Definition 4.1. *[13] Let (X, φ) and (Y, ψ) be dynamical systems. A* ***map*** $\pi : (X, \varphi) \to (Y, \psi)$ *is a continuous function such that $\pi \circ \psi = \varphi \circ \pi$. If π is also surjective, it is called a* ***factor map****.*

In our construction of an s/u-bijective pair for the tilings spaces we will consider, we will require the notion of a shift of finite type.

Definition 4.2. *[11] Let A be a finite alphabet (or set).*

- *The* ***full shift*** $A^{\mathbb{N}}$ *on a set A is the collection of all bi-infinite sequences of elements of A. Denote each element of the shift as $(x_i)_{i \in \mathbb{Z}}$.*

- *The* ***shift map*** σ *on a full shirt $A^{\mathbb{N}}$ maps a point $x \in A^{\mathbb{N}}$ to a point $y = \sigma(x)$ such that the $y_i = x_{i+1}$. This is conventionally thought of a shifting the origin one space to the left in the original sequence.*

- *A* ***shift of finite type*** *is a subset of a full shift where a finite number of blocks are forbidden from appearing in any sequence.*

Example 4.1. *An example of a shift of finite type would be all bi-infinite sequences made up of elements of the set* $\{1, 2\}$*, where the symbol 2 is not allowed to follow itself. Thus the sequence*

$$\ldots 12112121 \ldots$$

would be allowed but the sequence

$$\ldots 121122121 \ldots$$

would not.

Definition 4.3. *Let G be a graph and (Σ_G, σ) be the associated shift of finite type. Given a Smale space (X, φ) with Smale constant ϵ_X and a factor map $\pi : (\Sigma_G, \sigma) \to (X, \varphi)$, we say the factor map is* ***regular*** *if, for all $e, f \in \Sigma_G$ such that $t(e^0) = t(f^0)$, we have $d(\pi(e), \pi(f)) \leq \epsilon_X$ and $\pi[e, f] = [\pi(e), \pi(f)]$.*

That is, the map is regular if the Smale bracket is defined and the map commutes with the bracket operation. We note that any factor map can be made regular by taking the graph to a higher block presentation such that there is only one edge between each pair of vertices. For a full explanation of constructing a higher block presentation, with many examples, see Lind and Marcus [11].

Definition 4.4. *[13] Suppose that $(X, \varphi), (Y_1, \psi_1)$, and (Y_2, ψ_2) are dynamical systems and that $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$ are maps. The* ***fibered product*** *of Y_1 and Y_2 is the space $Z = \{(y_1, y_2) | y_1 \in Y_1, y_2 \in Y_2, \pi_1(y_1) = \pi_2(y_2)\}$.*

It is also possible to create the fibered product of a space with itself.

Definition 4.5. *[13] Let $\pi : (Y, \psi) \to (X, \varphi)$ be a map. For $N \geq 0$, define*

$$Y_N(\pi) = \{(y_0, y_1, \ldots, y_N) \in Y^{N+1} | \pi(y_i) = \pi(y_j) \text{ for all } 0 \leq i, j \leq N\}.$$

Definition 4.6. *[13] Let* (X, φ) *and* (Y, ψ) *be Smale spaces and let*

$$\pi : (Y, \psi) \to (X, \varphi)$$

be a map. The map π *is called s-**bijective (u-bijective)** if for any* y *in* Y*, the restriction of* π *to* $Y^s(y)$ *(*$Y^u(y)$*) is a bijection to* $X^s(\pi(y))$*(*$X^u(\pi(y))$*,respectively).*

Definition 4.7. *Let* (X, φ) *be a Smale space. We define an s/u-**bijective pair** to be the set* $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$*, where*

1. (Y, ψ) *and* (Z, ζ) *are Smale spaces.*
2. $\pi_s : (Y, \psi) \to (X, \varphi)$ *is an s-bijective factor map.*
3. $Y^u(y)$ *is totally disconnected for every* y *in* Y
4. $\pi_u : (Z, \zeta) \to (X, \varphi)$ *is a u-bijective factor map.*
5. $Z^s(z)$ *is totally disconnected for every* z *in* Z

Theorem 4.1. *[13] If* (X, φ) *is a non-wandering Smale space, there exists an s/u-bijective pair for* (X, φ)*.*

Theorem 4.2. *[13] The homology of a Smale space is independent of the choice of s/u-bijective pair.*

For a substitution tiling space, we may greatly simplify our s/u-bijective pair.

Theorem 4.3. *[13] Let* (X, φ) *be a non-wandering Smale space. Then there exists a shift of finite type* (Σ, σ) *and a factor map*

$$\pi : (\Sigma, \sigma) \to (X, \varphi)$$

such that π may be written as the composition of an s-bijective factor map with a u-bijective factor map.

Since a tiling space is non-wandering, and has a totally disconnected stable set, given a tiling space (X, φ), we may choose $\pi = (\Sigma, \sigma, \pi_s, X, \varphi, id)$ as our s/u-bijective pair, where id represents the identity map, that is, $Z = X$ and $\zeta = id$.. We are then left with the problem of how to construct this shift of finite type and the s-bijective factor map π_s. This will be resolved using the notion of a Markov partition.

Definition 4.8. *[11] A* ***topological partition*** *of a metric space M is a finite collection* $\mathcal{P} = \{P_0, P_1, \ldots, P_{r-1}\}$ *of disjoint open sets such that* $M = \overline{P_0} \cup \cdots \cup \overline{P}_{r-1}$.

We may construct a shift space $X_{\mathcal{P},\phi}$ whose elements each denote a member of the topological partition. Thus, if we have a dynamical system, a sequence in $X_{\mathcal{P},\phi}$ may record into which member of the partition a particular point falls at a particular iteration of the system. For every $x \in X_{\mathcal{P},\phi}$ we may define the non-empty set

$$D_n(x) = \bigcap_{k=-n}^{n} \phi^{-k}(P_{x_k}) \subseteq M.$$

Definition 4.9. *[11, 6] Let* (M, ϕ) *be an invertible dynamical system. A topological partition of M is a* ***Markov Partition*** *if for every* $x \in X_{\mathcal{P},\phi}$*, the intersection* $\bigcap_{n=0}^{\infty} \overline{D}_n(x)$ *consists of exactly one point and* $X_{\mathcal{P},\phi}$ *is a shift of finite type.*

In the next section, we show hot to construct a Markov partition for a substitution tiling space. We will utilize the existing coding given by the tiles in each tiling to simply record into which tiles the origin falls under substitution. It should be noted that the resulting Markov partition is a shift of finite type, whereas the coding from the tiles in a tiling is never a shift of finite type.

Chapter 5

Computing the homology of a substitution tiling space

In chapter 3 we discussed a cohomology theory that was entirely derived from the action of the substitution map. Here we present a homology theory that, while related to the substitution, requires us to look at other properties of the space. The homology theory was developed by Putnam in [13] for any general Smale space. We have adapted his method here to be specific to aperiodic substitution tiling spaces.

Let us first digress briefly to define and discuss the dimension groups of Krieger [9]. These will be the entries in our chain complex that allow us to compute the homology of the tiling space. There are two dimension groups, denoted stable and unstable, associated with any shift of finite type. We will only concern ourselves with the stable group.

Definition 5.1. *Given a shift of finite type* (Σ, σ), *let* $\mathcal{D}^s(\Sigma)$, *be the collection of all non-empty, compact, open subsets of* Σ, *with the following equivalence relation. Given* $E, F \in \Sigma$, *let* $E \sim F$ *if* $[E, F] = E$ *and* $[F, E] = F$, *with the restriction that* E F *if and only if* $\sigma(E)$ $\sigma(F)$. *Let* $[E]$ *be the equivalence class of* E. *The stable* ***dimension group***, *denoted* $D^s(\Sigma)$ *is defined to be the free abelian group generated by* $\mathcal{D}^s(\Sigma)$, *modulo the subgroup generated by all* $[E \cup F] - [E] - [F]$, *where* E *and* F *are disjoint.*

This definition is rather obtuse, so we will need a more direct way of computing the dimension groups. By choosing a graph which presents our shift of finite type [11, 13], we may compute a dimension group for the graph which is isomorphic to the dimension group of the shift of finite type. For our purposes, we will only require the stable group, which will be denoted with a superscript s. For the full development of both groups, see Chapter 3 of [13]. Let G be a graph and consider $\mathbb{Z}G^0$, the abelian group generated by the set of vertices, G^0. For any edge $e \in G$, let the maps $t(e)$ and $i(e)$ denote the vertex at which that edge terminates and originates, respectively. We define a map $\gamma_G^s : \mathbb{Z}G^0 \to \mathbb{Z}G^0$ by

$$\gamma_G^s(v) = \Sigma_{t(e)=v} i(e).$$

We then let the dimension group be defined by

$$D^s(G) = \varinjlim(\mathbb{Z}G^0, \gamma_G^s).$$

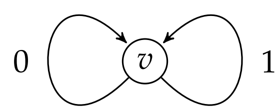
The standard definition of this direct limit is given in [10], but we use the alternate definition from [13].

Definition 5.2. *Construct the set $\mathbb{Z}G^0 \times \mathbb{N}$, and let $(a, n) \sim (b, n)$ if there exists some $l \geq 0$ such that $(\gamma_G^s)^{n+l}(a) = (\gamma_G^s)^{m+l}(b)$. For $(a, m) \in \mathbb{Z}G^0 \times \mathbb{N}$, let $[a, m]$ denote it's equivalence class. The* ***direct limit*** *can be seen as the set of all equivalence classes of this set.*

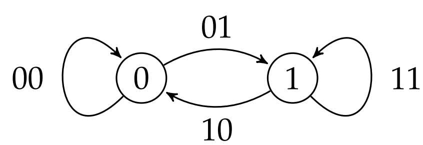
In practice, we will compute the direct limit by examining the adjacency matrix of the graph G. If we calculate the eigenvalues of this matrix, we may use these to see what group persists in the direct limit. Any non-zero eigenvalue plays a role. For each integer eigenvalue λ, we get one copy of the integers with $\frac{1}{\lambda}$ adjoined.

The direct limit is then the direct sum of these groups. For example, if the non-zero eigenvalues of the matrix were $1, -1, 2$, the direct limit would be $\mathbb{Z}^2 \oplus \mathbb{Z}[\frac{1}{2}]$.

We will also need the dimension group of the higher block presentations of a graph. These are constructed by letting the edge set of a graph be the vertex set of another graph. We look at the allowed transitions between edges, and set each of these transitions as a new edge. For example, we could have the graph G



with the vertex set $\{v\}$ and edge set $\{0, 1\}$, and it's higher block presentation G^1



with vertex set $\{0, 1\}$ and edge set $\{00, 01, 10, 11\}$. Note that the edges in G^1 are simply pairs of edges from G^0. We may continue in this manner to generate the higher block presentation G^K.

Theorem 5.1. *Let G be a graph and $K, K' \geq 0$. Then $D^s(G^K) \cong D^s(G^{K+K'})$.*

Now that we have the dimension group of a graph established, we require the dimension group of a shift of finite type. Let (Σ_G, σ) be the shift of finite type associated to the graph G. Then (Σ_G, σ) is the collection of all bi-infinite paths in G. We fortunately have the result of Putnam that $D^s(G) \cong D^s(\Sigma_G, \sigma)$.

We will need to compute higher dimensional dimension groups to build our chain complex. For these we will look at higher dimensional graphs, using the fibered product. Let G_n denote the fibered product of G with itself n times. Then G_n^k is the set of all paths of length $k+1$ in this fibered product.

Definition 5.3. *[13] Let $k, N \geq 0$.*

- *Let $\mathcal{B}(G_N^k, S_{N+1}) \subseteq \mathbb{Z}G_N^k$ be the subgroup generated by all elements $p \in G_N^k$ such that $p = p \cdot \alpha$ and all elements $p = sgn(\alpha)p \cdot \alpha$, for some permutation $\alpha \in S_{N+1}$.*
- *Let $\mathcal{Q}(G_N^k, S_{N+1})$ be the group*

$$\mathbb{Z}G_N^k / \mathcal{B}(G_N^k, S_{N+1}).$$

Example 5.1. *Using the second graph above, let G_1^0 be the paths of length 1 in the fibered product. We then have $G_1^0 = \{(0,0), (0,1), (1,0), (1,1)\}$. We look at the action of elements of this set under the group S_2. There are only two elements of this permutation group, one of which is the identity, and the other of which swaps the entries of an element of G_1^0. If we look at the elements that are equal to themselves under a permutation we have $(0,0)$ and $(1,1)$. The elements $(0,1)$ and $(1,0)$ are permutations of each other, so we only consider one of them when we generate $\mathcal{B}(G_1^0, S_2) \cong \mathbb{Z}^3$. Therefore $\mathcal{Q}(G_1^0, S_2) \cong \mathbb{Z}^4/\mathbb{Z}^3 \cong \mathbb{Z}$. We will show a method later in this chapter for determining the final group $\mathcal{Q}(G_N^k, S_{N+1})$ more quickly.*

Definition 5.4. *The higher dimensional Krieger dimension groups are defined by*

$$D_{\mathcal{Q}}^s(G_N^k) = \varinjlim(\mathcal{Q}(G_N^k, S_{N+1}), \gamma^s).$$

To actually build our chain complex, we begin with a substitution tiling space with finite local complexity, which we denote (X, φ), where X is our space and φ is the substitution map.

From this we build an s/u-bijective pair, letting (Y, ψ) be a Markov partition on the tiling space (the construction will be described shortly), and $(Z, \zeta) = (X, \varphi)$, with ζ the identity map. The requirements for and s/u-bijective pair are met, as the identity map is bijective, and therefore u-bijective, and the stable sets of a substitution tiling space are totally disconnected, satisfying the condition for (Z, ζ). For our space (Y, ψ), we require the map $\pi^s : Y \to X$ to be a regular, s-bijective factor map. By choosing our map to be

$$\pi^s(y) = \bigcap_{n=-\infty}^{\infty} \varphi^{-n}(y_n)$$

as in [6], these conditions are satisfied.

In building our Markov partition, we code the each point in our tiling space by recording the prototile type into which the origin falls under each iteration of the substitution. In the case that the origin lands on the border of two tiles, we make a choice to code by the left or right tile. Once this choice has been made, the same choice must be made for the remainder of the tiling. The one concern in making this coding is that, once the tiling are coded, the substitution map must be invertible, for which we need recognizability, as defined in [14]. We basically need to ask if one can determine the pre-image of a particular tile under the substitution map. This is essential given the factor map above involves inverting the substitution. Recognizability can be obtained by taking the j-th order collaring, that is, let each ”rectangle” in the partition be selected to be the collared prototile containing the origin, with the collaring chosen so the j nearest neighbors of this tile are known. For some such j, the tiling is always recognizable.

Now that we have our Markov partition, we note that it is clearly a shift of finite type. Therefore, we may apply the following theorem from [13] to compute the homology of the tiling space.

Theorem 5.2. *Let (X, φ) be a Smale space and (Σ, σ) a shift of finite type. Suppose that $\pi_s : (\Sigma, \sigma) \to (X, \varphi)$ is an s-bijective factor map. Then the homology $H^s_N(X, \varphi)$ is naturally isomorphic to the homology of the complex $(D^s_Q(\Sigma_*(\pi_s)), d^s(\pi_s))$*

Relabeling our Markov partition as $(Y, \psi) = (\Sigma, \sigma)$, we need to find the complex $(D^s_Q(\Sigma_*(\pi^s)), d^{s,K}(\pi^s))$ and compute its homology. In coding our space into the Markov partition, we made a choice of left or right tile when the origin landed on the border of two tiles. Thus we may have two different codings that map to the same tiling.

Theorem 5.3. *[13] Given an s/u-bijective pair for a Smale space (X, φ), let M_0 be such that the cardinality of $\pi^s\{x\}$ is less than M_0 for all $x \in X$. Then for $M \geq M_0$, the dimension group $D^s_M = 0$.*

Our factor map here is $2:1$, and by the theorem above, we need to only consider the (0,0) and (1,0) entries in our complex.

Begin by building a graph G to represent the Markov Partition under the substitution map. In order for the factor map to be regular, we must have only one edge connecting each pair of vertices. If this is not the case, taking a higher block presentation of the graph will remedy the situation. We denote it's vertex set by G^0 and it's edge set by G^1. Since $D^s_Q(\Sigma_0) \cong D^s_Q(G_0)$, we need only consider the dimension group of the graph. Let A be the adjacency matrix of the vertices of G_0, and take $\varinjlim(\mathbb{Z}G_0^0, A)$. This will yield the (0,0) entry of our complex.

To compute the next element of our chain complex, we need to define G_1 be be the fibered product of G with itself. It is defined as $G_1 = \{(x_0, x_1) \in G \times G | \pi^s(x_0) = \pi^s(x_1)\}$. We take the vertex set of this fibered product and denote if G_1^0.

Definition 5.5. *Given a permutation group S_n and a set X on which it is acting, the* ***isotropy subgroup*** *of S_n at an element $x \in X$ is the set of all $\alpha \in S_n$ such that $x \cdot \alpha = x$. We say that the element x has* ***trivial isotropy*** *if the isotropy subgroup at this point consists of only the identity element.*

To compute $D_Q^s(\Sigma_1)$, we need to find a subset $B_1^0 \subseteq G_1^0$, which contains only elements meeting each orbit having trivial isotropy only once, and not meeting any orbits having non-trivial isotropy under action by S_2. We will eventually take the direct limit of this the group generated by this set, but we must first define the bonding maps involved. Let $t_B^*(p, j) = \{(q, \alpha) \in G_1^1 \times S_2 | t^j(q) = p, i^j(q) \cdot \alpha \in B_1^0\}$. Then we may define

$$\gamma_B^s(p) = \sum_{(q,\alpha) \in t_B^*(p,1)} \operatorname{sgn} \alpha \cdot i(q) \cdot \alpha.$$

Let A' be the adjacency matrix of the vertices of B_1, as defined by γ_B^s, and take $\varinjlim(\mathbb{Z}B_1^0, A')$. This gives us $D_Q^s(G_1)$ and the $(1, 0)$ entry of our complex.

We must state one important theorem before we proceed with our computation.

Theorem 5.4. *[13] Let (Σ, σ) be a shift of finite type, (X, φ) be a Smale space and π^s an s-bijective factor map. Then $D_Q^s(\Sigma_N(\pi^s), d_Q^s(\pi^s)_N)$ is a chain complex.*

We now introduce notation for our boundary maps. The complex we will use for our one-dimensional tiling spaces is

$$0 \to D^s_Q(\Sigma_1) \xrightarrow{d_1^{s,K}} D^s_Q(\Sigma_0) \xrightarrow{d_0^{s,K}} 0.$$

As per [13], $d_0^{s,K}$ is defined to be the zero map. We need then only to define $d_1^{s,K}$ to finish building our chain complex.

We define $d_n^{s,K}(q) = \sum_{j=1}^{n}(-1)^n \delta_n^K(q)$, where $q \in G_n$. Thus, in our case, we have $d_1^{s,K}(q) = \delta_0^K(q) - \delta_1^K(q)$. It leaves us then to formally define the δ maps.

The map δ_n takes in an $n-$tuple in G_n and deletes the $n-$th coordinate. We define the map

$$\delta_n^K(q) = Sum\{\delta_n(q')|q' \in G_1^{1+K}, t^K(q') = q\}.$$

In other words, we want to list all paths in G_1 of length $1 + K$ that terminate in the ordered pair q, delete the $n-$th coordinate, then sum the resulting set. The first obstacle here is in finding K. It is imperative that we choose K sufficiently large. As demonstrated in the examples in Chapter 6, an incorrect choice of K will result in an incorrect computation.

Using Lemma 2.7.2 in [13], we put a constraint on K. Given bi-infiite paths $e_0, e_1, f_0, f_1 \in (\Sigma, \sigma)$, and a constant k_0 such that $\pi^s(e_0) = \pi^s(e_1)$, $\pi^s(f_0) = \pi^s(f_1)$, $e_0^k = f_0^k$ for all $k \geq k_0$, and e_1 stably equivalent to f_1, we have $e_1^k = f_1^k$ for all $k \geq k_o + K$. Any K satisfying this condition may be used in our definition of δ_n^K above. For a one-dimensional substitution tiling and our chosen Markov partition, we need to consider how many times two tilings may disagree near the origin if they are to agree everywhere thereafter. This is essentially equivalent to knowing the asymptotic composants of a tiling space as described in [2]. There Barge and Diamond present an algorithm for computing the asymptotic composants of a tiling space.

Once this is complete, we are ready to compute our boundary map. We first look at the generators of $D^s_Q(\Sigma_1)$. Using the eigenvectors which correspond to the eigenvalues we used to compute this direct limit, we identify the equivalence classes into which the elements of B^0_1 were separated. By theorem 3.4.4 in [13], we know that the image of the equivalence classes under δ_n is the same as if we apply $\delta^{s,K}_n$ to the individual generators and take their equivalence classes. That is, if we let $[g] \in D^s_Q(\Sigma_n)$ represent the equivalence class of an element of Σ_n, we have

$$\delta_n([g]) = [\delta^{s,K}_n(g)].$$

Since $\varinjlim(\mathbb{Z}G^0_0, A) \cong \varinjlim(\mathbb{Z}G^K_0, A)$ by theorem 3.2.3 in [13], we can look to see which eigenspaces the generators of $D^s_Q(\Sigma_1)$ end up in after applying the d_1 map. This allows us to identify the kernel and image of the d_1 map and thus compute the homology.

Chapter 6

Examples

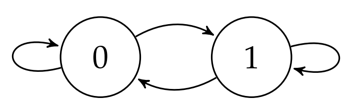
Our first example is the 2-solenoid. While not technically a tiling space, it has very similar properties, being the inverse limit of the doubling map on the circles. It illustrates that our homology computation is independent of the choice of Markov partition.

We first compute it's cohomology. This can be done using the method described in Theorem 73.4 of Munkres [12] for computing the Čech cohomology of an inverse limit space. In particular, let X be the 2-solenoid, S^1 the unit circle, and φ the doubling map, then

$$\check{H}^1(X) \cong \check{H}^1(\varprojlim(S^1, \varphi)) \cong \varinjlim(H^1(S^1), \varphi) \cong \varinjlim(\mathbb{Z}, \varphi).$$

We have that, for the 2-solenoid, the cohomology groups are $\check{H}^0 \cong \mathbb{Z}$ and $\check{H}^1 \cong \mathbb{Z}[\frac{1}{2}]$.

The method for computing the homology of the solenoid comes from [13]. We first look at the Markov partition of the solenoid with two elements. The graph in this case is



and the adjacency matrix is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, which has eigenvalues 0 and 2. Thus $D^s_Q(G_0) \cong \mathbb{Z}[\frac{1}{2}]$. To compute $D^s_Q(G_1)$, we consider the set $G^0_1 = \{(0,0), (1,1), (0,1), (1,0)\}$

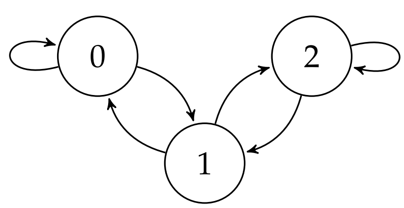
and it's subset $B_1^0 = \{(0,1)\}$. Since B_1^0 has only 1 element, the associated direct limit is $\mathbb{Z}$. This gives us a chain complex of

$$0 \to \mathbb{Z} \xrightarrow{d_1} \mathbb{Z}[\frac{1}{2}] \to 0.$$

Applying the boundary map, we have $d[(0,1)] = [\delta_0^{s,K}(0,1)] - [\delta_1^{s,K}(0,1)]$. These appear to be different equivalence classes, but since there is only one equivalence class in the range, they must go to the same class. Therefore the boundary map is the zero map and we have

$$H_0 \cong \mathbb{Z}[\frac{1}{2}] \cong \check{H}^1$$
$$H_1 \cong \mathbb{Z} \cong \check{H}^0.$$

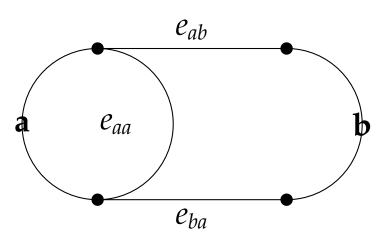
It is important to note that the homology is independent of the choice of Markov partition. If we use 3 rectangles in our partition, instead of 2, we get the same result. Looking at the solenoid as being created by a map that wraps a circle around itself twice, we can partition the circle into 3 equal pieces and see where they are mapped in the wrapping. The first third of the circle would wrap around the first two thirds, the second third would wrap around the last and first third, etc. This gives us this graph



The adjacency matrix is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, which has eigenvalues $\lambda = -1, 1, 2$ and thus $D^s_Q(G_0) \cong \mathbb{Z}^2 \oplus \mathbb{Z}[\frac{1}{2}]$.

$G^0_1 = \{(0,0),(1,1),(2,2),(0,1),(0,2),(1,2),(2,1),(1,0),(2,0)\}$ and the subset $B^0_1 = \{(0,1),(0,2),(1,2)\}$ can be found to have the adjacency matrix $A' = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$. The direct limit gives us $D^s_Q(G_1) \cong \mathbb{Z}^3$. If we take the three eigenvectors, and apply the boundary map, we see that one of them is clearly zero, while the other two are both non-zero, and do not share an eigenspace. Thus the kernel of the boundary map is $\mathbb{Z}$ and the image $\mathbb{Z}^2$, which yields the same homology computation as above.

Next we consider the Fibonacci Tiling space. The space is generated by the substitutions $a \to ab$ and $b \to a$. We begin by computing the cohomology using the Barge-Diamond method. Our CW complex, including germs, is

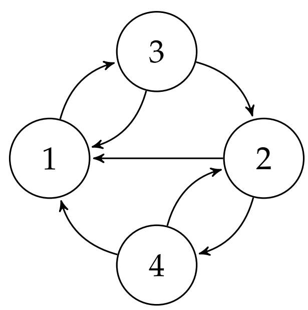


The adjacency matrix is $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. There is only one element in the eventual range, so there are no loops and only one connected component. The direct limit is just $\mathbb{Z}^2$, so we have $\check{H}^1 \cong \mathbb{Z}^0 \oplus \mathbb{Z}^2/\mathbb{Z}^0 \cong \mathbb{Z}^2$

For the homology computation, our Markov coding is achieved by placing the location of the origin at each iteration of the substitution into one of the following four rectangles

$$
\begin{aligned}
1 &: \quad (a)b(a) \\
2 &: \quad (b)a(a) \\
3 &: \quad (a)a(b) \\
4 &: \quad (b)a(b)
\end{aligned}
$$

with the origin landing on the border handled as above. This gives us a Markov partition which is actually a shift of finite type. Therefore, the Krieger dimension groups for this shift of finite type are isomorphic to those generated by the graph which presents the shift. We use this graph, which we label G.



Let $G^0 = \{1,2,3,4\}$ be the vertex set of G, and $G^1 = \{12,21,23,31,34,41,43\}$ be the edge set of G. We will first compute $D^s_Q(G_n)$, for $n = 0,1$. This is quite easy in the case $n = 0$, as our group is simply the direct limit of $\mathbb{Z}G^0_0$, under the map

$$\gamma^s(e) = \sum_{t(e)=v} i(e),$$

where $v \in G^0, e \in G^1$, and i and t represent the initial and terminal maps respectively. This is the same map as the adjacency matrix $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. As there are 4 elements of G_0^0, we have our actual limit as $\varinjlim(\mathbb{Z}^4, A)$. Since the eigenvalues of A are $0, -1, \frac{1 \pm \sqrt{5}}{2}$, we have that this direct limit is isomorphic to $\mathbb{Z}^3$

In the case $n = 1$, our procedure becomes more complicated. We will still be taking the direct limit of a group generated by the graph, but calculating the group and the map is not quite as straightforward. We can list all elements of $G_1^0 = \{(1,1),(2,2),(3,3),(4,4),(1,2),(1,3),(1,4),(2,1),(3,1),(4,1),(3,2),(2,3)\}$, and thus $B_1^0 = \{(1,2),(1,3),(1,4),(2,3)\}$. Computing the map γ_B^s,and applying this map to each $p \in B_1^0$, we have the adjacency matrix $A' = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix}$. Thus our group in question is $\varinjlim(\mathbb{Z}^4, A') \cong \mathbb{Z}^2$.

This gives us a chain complex of

$$0 \to \mathbb{Z}^2 \xrightarrow{d_1} \mathbb{Z}^3 \to 0.$$

To compute the boundary map, we let $K = 2$. If $K < 2$ is chosen, the computation may still be carried out, but the result will be incorrect. For too small of a K the $\delta_n^{s,K}$ map will not commute with the generators of the equivalence classes, resulting in an incorrect boundary map. We then list all paths of length 3 in our original graph, of which there are 7. We then take the eigenvectors associated with each eigenspace that generated $\mathbb{Z}^2$ and augment it to the 7×7 matrix of the eigenvectors generating $\mathbb{Z}^3$. If we row reduce this matrix, we see that one of the generators is a linear combination of the eigenvectors associated to non-zero eigenvalues, while the other is not. Thus the image and kernel of the boundary map are each $\mathbb{Z}$, giving us that the homology is

$$H_0 \cong \mathbb{Z}^2 \cong \check{H}^1$$

$$H_1 \cong \mathbb{Z} \cong \check{H}^0.$$

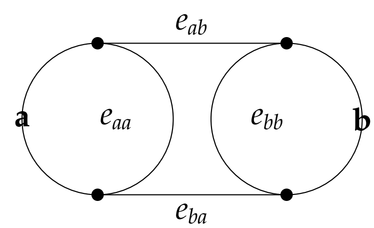
For our next example, we consider the tiling space generated by the rules $a \to aaabb$ and $b \to ab$. The cohomology for this example was computed in Chapter 3.

When we look at the possibilities for neighbored tiles, there are 7 of them. In this case it becomes easier to represent the possible paths as an edge list, rather than as a graph. This gives us $G_0^1 = \{11, 12, 13, 14, 15, 21, 22, 23, 24, 25, 31, 32, 33, 34, 35, 46, 47, 51, 52, 53, 54, 55, 61, 62, 63, 64, 65, 76, 77\}$ and adjacency matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$.
Taking the direct limit yields $D_Q^s(G_0) \cong \mathbb{Z}^2$. Moving up one level, we have the set $B_1^0 = \{(1,2), (2,3), (3,4), (4,5), (5,1), (4,6),$
$(6,7), (7,5)\}$, with the direct limit being $\mathbb{Z}$. Taking the elements that generate this copy of the integers and applying the boundary map leaves a single vector that is in the zero eigenspace of A. Thus the boundary map is zero, and we have

$$H_0 \cong \mathbb{Z}^2 \cong \check{H}^1$$

$$H_1 \cong \mathbb{Z} \cong \check{H}^0.$$

Finally, we compute the homology for the Morse-Thue tiling space, generated by $a \to ab$ and $b \to ba$. The cohomology is calculated using the CW complex (including germs)



where all 4 germs are in the eventual range. Thus we have 1 connected component and one loop. The direct limit of the adjacency matrix is $\mathbb{Z}[\frac{1}{2}]$, so we have $\check{H}^1 \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}]$.

For the homology computation we need to know 2 neighbors of each tile to have recognizability. This gives us 12 rectangles in our Markov partition, and 16 elements in the set B_1^0. The chain complex becomes

$$0 \to \mathbb{Z}^4 \xrightarrow{d_1} \mathbb{Z}^4 \oplus \mathbb{Z}[\frac{1}{2}] \to 0.$$

We must take $K \geq 2$, which results in a 48×48 matrix, which we omit. The kernel of the boundary map can be computed to be $\mathbb{Z}$, which gives us, correctly,

$$H_0 \cong \mathbb{Z} \oplus \mathbb{Z}[\frac{1}{2}] \cong \check{H}^1$$

$$H_1 \cong \mathbb{Z} \cong \check{H}^0.$$

Chapter 7

Conclusion

In this paper we have presented several examples supporting that, for all one-dimensional substitution tiling spaces Ω, the homology and Čech cohomology are related via

$$H_0(\Omega) \cong \check{H}^1(\Omega) \text{ and } H_1(\Omega) \cong \check{H}^0(\Omega).$$

The key feature that leads us to believe that this is true are the asymptotic composants of a tiling space. According to Barge and Diamond [2], there is a finite, non-empty set of arc components of a tiling space that are called the asympototic composants. There is an algorithm for computing these given in the paper cited above. In our method for computing the homology of a tiling space, we looked at path of length K in the graph which presented our Markov partition. It seems to be the case when when, given two elements of our group $\mathbb{Z}B_1^K$ where the first coordinate of one element is equal to the second coordinate of the other element, these will end up in the kernel of the boundary map. This suggests that only those arc components which are asymptotic composants will be in the image. This leads us to believe that, as the cohomology and homology of an aperiodic substitution tiling space are each related to the asympototic composants of the space, we will find the duality of this conjecture.

Unfortunately, there has been some difficulty in obtaining an analog for the asymptotic composants of higher dimensional tiling spaces. There can be infinitely many asymptotics pairs, although finitely many in each direction. A recent paper by Barge and Olimb [5] offers some hope, but little in the way of algorithmically computing asymptotics. Further work would be needed in this area before the general conjecture could be pursued in higher dimensions.

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