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A Comparison of Smale Spaces and Laminations via Inverse Limits

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by

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ABSTRACT

Inverse limits began as a purely topological concept, but have since been applied to areas such as dynamical systems and manifold theory. R.F. Williams related inverse limits to dynamical systems by presenting a construction and realization result relating expanding attractors to inverse limits of branched manifolds. Wieler then adapted these results for Smale Spaces with totally disconnected local stable sets. Rojo used tiling space results to relate inverse limits of branched manifolds to codimension zero laminations. This paper examines the results of Wieler and Rojo and shows that they are analogous.

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Chapter 1

Introduction

Historically, inverse limits arose as a purely topological concept. The most famous example of an inverse limit is a solenoid, which was defined in 1927 by L. Vietrois ([Vie27]). It seems a simple concept, but as Anderson and Choquet showed in 1959 ([AC59]), inverse limits are extremely useful in describing complicated spaces by producing them from simpler ones. For instance, G. W. Henderson showed that the pseudo-arc is the inverse limit on [0,1] with a single bonding map ([KO10]). R.F. Williams was responsible for connecting dynamical systems. In 1967, he applied them to non wandering sets, and in 1974, he applied them to attractors ([Wil67], [Wil74]). Inverse limits are now commonly used in the field of dynamical systems, and will be examined further here.

In 1967, Smale defined a class of dynamical systems known as Axiom A, which is defined in terms of the non wandering set and the periodic points ([Sma66]). Smale's Spectral Decomposition Theorem stated that the non-wandering set can be written as the finite union of basic sets. In 1978, Ruelle examined the topological dynamics of the basic sets, and in so doing defined Smale Spaces ([Rue04]). Since then, there have been many results regarding these spaces. For a summary, see Putnam ([Put12]). Of particular interest for us will be the work of Wieler, who generalized some of Williams' results for Smale spaces ([Wie12a]). These results are discussed thoroughly in Chapter 5.

Inverse limits have also been used for results regarding tiling spaces. In 2006, Bellisard, Benedetti and Gambaudo proved that the continuous hull of any aperiodic and repetitive planar tiling is an inverse limit of branched flat surfaces ([BBG06]). In [CRS11], Cuesta, Rojo and Stadler showed that any minimal transversely Cantor compact lamination is an inverse limit of branched manifolds. This paper also had the useful mindset of thinking of laminations in terms of tiling spaces. In 2012, Rojo adapted these results for codimension zero laminations, which gives us the second focus of this paper ([LR13]).

The aim of this paper is to relate the ideas of inverse limits, Smale spaces and laminations. Wieler and Rojo both adapted results from [Wil74]. Wieler generalized the results for Smale spaces and Rojo looked at laminations, but they are built off the same starting point and will be shown to be analogous.

Because they adapted Williams' result to different topics, we will need to introduce those topics before we discuss them together. Chapters 2, 3 and 4 will discuss inverse limits, Smale spaces and laminations, respectively. Each of those will include terminology, examples and proofs of some basic results. Chapter 5 will discuss the theorems by Wieler and Rojo, and will compare them.

Chapter 2

Inverse Limits

We will see that inverse limits are a powerful tool for explaining the behavior of hyperbolic dynamical systems, but first it is necessary to build up our terminology and understanding of the concepts separately. We will begin with defining some basic topological definitions.

2.1 Definitions

A *product space* can be written $\prod_{i=0}^{\infty} X_i = \{(x_0, x_1, \cdots) : x_i \in X_i\}$, where each X_i is a topological space. The *product topology* associated with the product space has a basis of all sets of the form $\prod U_i$, where U_i is open in X_i for each i and U_i equals X_i for all but finitely many values of i. Given continuous maps $f_i : X_i \to X_{i-1}$, the *inverse limit space* on X_i is defined as $X_\infty = \varprojlim(X_i, f_i) = \{(x_0, x_1, \cdots) \in \prod_{i=0}^{\infty} |x_{i-1} = f_i(x_i)\}$. The functions f_i are commonly referred to as bonding maps, and the family of spaces (X_i, f_i) are sometimes referred to as a *projective system*. The inverse limit space is a metrizeable subspace of the product space which inherits the topology of the product space. This is proved in Proposition 2.3.3. Another useful function is the *projection map* $\pi_\beta : \prod_{\alpha \in \mathcal{A}} X_\alpha \to X_\beta$, which maps an element of a Cartesian product space to the β^{th} factor space. Projection maps are both continuous and open, the latter meaning that open sets are mapped to open sets.

If $\{a_i\}_{i\in\mathbb{N}} \subset \mathbb{N}$ is an infinite set, then a *telescoping contraction* of the projective system (X_k, f_k) is the projective system (X_{a_i}, g_k) where $g_k : X_k \to X_{k-1}$ is defined as

$$g_i = f_{a_i} \circ \cdots \circ f_{a_{i-1}+2} \circ f_{a_{i-1}+1}.$$

The map $f : X_i \to X_{i-1}$ is *flattening* if for each $x \in X_i$ there exists a (normal) neighborhood U such that $f(U)$ is a smooth disk of X_{i-1}. A projective system is *flattening* if there is a telescopic contraction of it with each f_i flattening. The inverse limit of the telescoping system is homeomorphic to the original inverse limit since they are cofinal to each other ([Mun84]). This idea appears in Theorem 5.2.1 which connects inverse limit spaces to laminations.

2.2 Examples

Example 2.2.1. *Perhaps the most famous example of an inverse limit is the solenoid. To construct the solenoid, start with a solid torus, $\mathbb{T}^2$. Inside this torus, we wrap another torus twice. Continuing in this manner, the solenoid is realized as the nested intersection of the tori, which is embedded in $\mathbb{R}^3$. This geometric construction of the solenoid is homeomorphic to the inverse limit $\varprojlim(S_i, f_i)$, where each S_i is the unit circle and f_i is the doubling map, which wraps S_{i+1} twice around S_i. Even though the solenoid is embedded in $\mathbb{R}^3$, it is only one-dimensional. See Figure 2.1.*

R.F. Williams defined a generalized n-solenoid in the following manner ([Wil74]). From here on, this solenoid will be referred to as a Williams solenoid, to differentiate it from the standard solenoid.

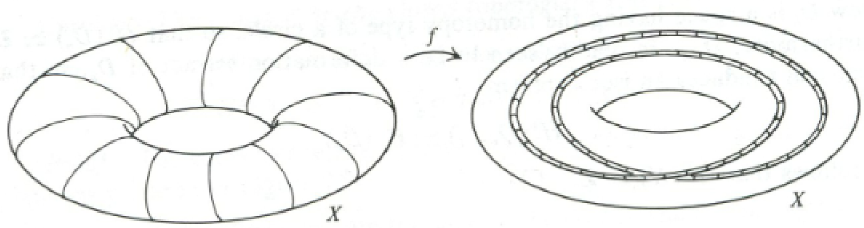


Figure 2.1: From [Mun84].

Definition 2.2.2. *Let K be a compact branched C^r n-manifold (sometimes called the Anderson-Putnam complex) and $g : K \to K$ a C^r immersion that satisfy the following three axioms.*

1) the non-wandering set of g is the entire space K.

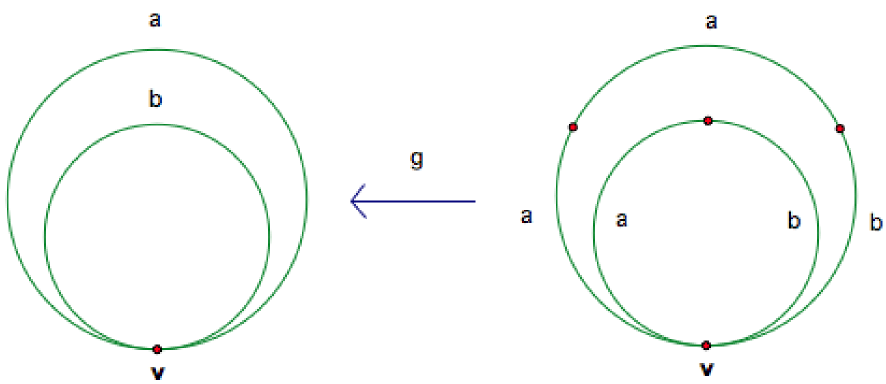
2) for each $x \in K$, there is a neighborhood N of x and $j \in \mathbb{Z}$ such that $g^j(N)$ is contained in a subset diffeomorphic to an open ball in $\mathbb{R}^n$.

3) g is an expansion. That is, there exist constants $A > 0$ and $\mu > 1$ such that, for all $n \in \mathbb{N}$ and $k \in T(K)$, $|Dg^n(k)| \geq A\mu^n|k|$, where $T(K)$ is the tangent space of K and Dg is the derivative of g.

A Williams n-solenoid is the inverse limit $\varprojlim(K, g)$.

The second axiom can be thought of as a flattening of a neighborhood of x. To see this more clearly, we can look at a Williams 1-solenoid. For $n = 1$, the second axiom requires that each point of K have a neighborhood whose image under g is an arc ([Wil67]).

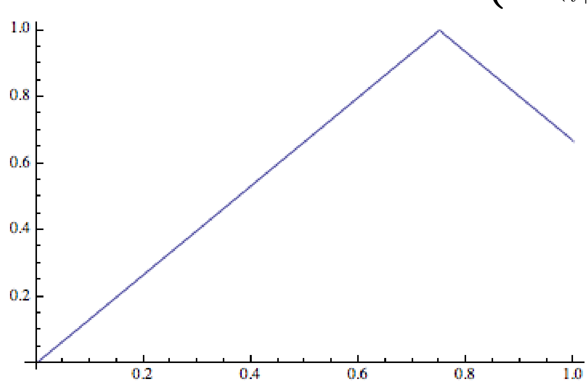
Example 2.2.3. *Let the space X be two circles, a and b, joined by a point v. Let $g : X \to X$ be defined by the wrapping maps $a \mapsto aab$ and $b \mapsto ab$.*



If $k = 1$, there is a neighborhood, V, of v, such that $g^k(V)$ is homeomorphic to an open interval, so g 'flattens' the space at the vertex. This flattening idea will be examined further in Chapter 5. Additionally, the map g actually defines a tiling of $\mathbb{R}$ as well, and is generated by the substitution rule on a and b.

Example 2.2.4. *The next example shows that the inverse of a simple map can be a complicated continuum.*

Let $X_i = [0,1]$ and $f_i : X_i \to X_i$ be defined by $f(x) = \begin{cases} \left(\frac{i+2}{i+1}\right) x & 0 \leq x \leq \frac{i+1}{i+2} \\ -\left(\frac{i+2}{i+1}\right) x + 2 & \frac{i+1}{i+2} \leq x \leq 1 \end{cases}$.



$\varprojlim(X_i, f_i)$ *is homeomorphic to the curve* $S = \left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : 0 < x \leq 1\right\} \cup (0,0)$, *which is known as the Topologist's Sine Curve, shown in Figure 2.2.*

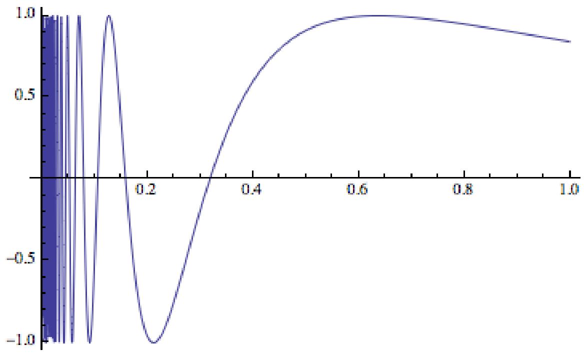


Figure 2.2: Topologist's Sine Curve

We will see in the next chapter that by changing the tent map slightly, the inverse limit is far from being a Smale space.

2.3 Basic Results

There are some commonly known results that show that, if we have information about each X_i, then we know properties of the inverse limit space. The following are sufficient conditions for $\varprojlim(X_i, f_i)$ being a continuum (a Hausdorff, compact, connected metric space).

Proposition 2.3.1. *If $\{X_i\}_{i=0}^{\infty}$ is a sequence of connected spaces, then $\varprojlim(X_i, f_i)$ is connected.*

Proof. Let each X_i be a connected, topological space and define $M_\beta = \{x \in \prod X_i | x_{i-1} = f_i(x_i)$ for all $i \leq \beta\}$. $M_1 = \mathcal{R}(f_1) \times \prod_{i=2}^{\infty} X_i$, $M_2 = \mathcal{R}(\mathcal{R}(f_2)) \times \mathcal{R}(f_2) \times \prod_{i=3}^{\infty} X_i$, etc. The continuous image of a connected set is connected, so $\mathcal{R}(f_1), \mathcal{R}(\mathcal{R}(f_2)) \cdots$, are connected sets. Also, the product space of connected spaces is connected, so each M_β is connected. $\cap_{\beta=1}^{\infty} M_\beta = \varprojlim(X_i, f_i)$, so $\varprojlim(X_i, f_i)$ is connected. ■

Lemma 2.3.2. *Given a metric* $d : X \times X \to \mathbb{R}$, $\overline{d(x,y)} = \min\{1, d(x,y)\}$ *is also a metric.*

Proof. Let X be a topological space with metric d. Define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d(x,y)} = \min\{1, d(x,y)\}$. Since d is nonnegative and symmetric, so is $\overline{d}$. If $x, y \in X$, it is also clear that $\overline{d(x,y)} = 0$ if and only if $x = y$.

To prove the triangle inequality, let $x, y, z \in X$, and look at two separate cases.

Case 1: $(d(x,y) \leq 1)$ In this case, $\overline{d(x,y)} = d(x,y)$. If $d(x,z)$ and $d(z,y)$ are both less than $\overline{d(x,y)} = d(x,y) \leq d(x,z) + d(z,y) = \overline{d(x,z)} + \overline{d(z,y)}$.

If $d(x,z)$ and $d(z,y)$ are both greater than 1, then $\overline{d(x,z)} + \overline{d(z,y)} = 2 > 1 \geq d(x,y) = \overline{d(x,y)}$.

The last subcase would be if only one of these is greater than 1. WLOG, suppose $d(x,z) \leq 1$ and $d(z,y) > 1$. In this case, $\overline{d(x,y)} = d(x,y) \leq 1 \leq \overline{d(x,z)} + 1 = \overline{d(x,z)} + \overline{d(z,y)}$.

Case 2: $(d(x,y) > 1)$ In this case, $\overline{d(x,y)} = 1 < d(x,y)$. If $d(x,z)$ and $d(z,y)$ are both less than or equal to 1, then $\overline{d(x,y)} = 1 < d(x,y) \leq d(x,z) + d(z,y) = \overline{d(x,z)} + \overline{d(z,y)}$.

If $d(x,z)$ and $d(z,y)$ are both greater than 1, then $\overline{d(x,y)} = 1 < 2 = \overline{d(x,z)} + \overline{d(z,y)}$.

Again, the last sub case would be if only one of these is greater than 1, so we suppose, WLOG, that $d(x,z) \leq 1$ and $d(z,y) > 1$. In this case, $\overline{d(x,y)} = 1 = \overline{d(z,y)} \leq \overline{d(x,z)} + \overline{d(z,y)}$. Therefore, in any case, $\overline{d(x,y)} \leq \overline{d(x,z)} + \overline{d(z,y)}$, so the triangle inequality holds, and $\overline{d}$ is a metric.

■

Proposition 2.3.3. *If $\{X_i\}_{i=1}^{\infty}$ is a sequence of metric spaces, then $\varprojlim(X_i, f_i)$ is a metric space with the subspace topology inherited from the product topology.*

Proof. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of metric spaces, each with associated metric d_i. Define $d : X_\infty \times X_\infty \to \mathbb{R}$ by $d(x, y) = \sum_{i=0}^{\infty} \frac{\overline{d_i(x_i,y_i)}}{2^i}$, where $\overline{d_i(x_i, y_i)} = \min\{1, d_i(x_i, y_i)\}$. We will show that d is a metric and that it induces the product topology on X_∞.

Clearly, d is non-negative and symmetric. Let $x, y \in X_\infty$. If $x = y$, then $d_i(x_i, y_i) = 0$ for all i and so $d(x, y) = \sum_{i=1}^{\infty} \frac{\overline{d_i(x_i,y_i)}}{2^i} = 0$. If $d(x, y) = 0$, then $\sum_{i=1}^{\infty} \frac{\overline{d_i(x_i,y_i)}}{2^i} = 0$. This implies $d_i(x_i, y_i) = 0$ for all i, which is only true if $x_i = y_i$ for all i, and so $x = y$. Therefore, $d(x, y) = 0$ IFF $x = y$. To prove that the triangle inequality holds, let $x, y, z \in X_\infty$.

$$\begin{aligned}
d(x, z) + d(z, y) &= \sum_{i=1}^{\infty} \frac{\overline{d_i(x_i, z_i)}}{2^i} + \sum_{i=1}^{\infty} \frac{\overline{d_i(z_i, y_i)}}{2^i} \\
&= \sum_{i=1}^{\infty} \frac{\overline{d_i(x_i, z_i)} + \overline{d_i(z_i, y_i)}}{2^i} \\
&\geq \sum_{i=1}^{\infty} \frac{\overline{d_i(x_i, y_i)}}{2^i} = d(x, y)
\end{aligned}$$

The above calculation made use of the fact that the triangle inequality holds for $\overline{d_i(x_i, y_i)}$, since it is itself a metric. The triangle inequality holds, so d is a metric.

To show that the metric induces the product topology, we will show that a set is open in X_i under d_i for all i if and only if it is open in $\varprojlim(X_i, f_i)$. Let B be a basic open element in the product topology. Then, we can write $B = \prod_{j=1}^{\infty} U_j$ where U_j is open in X_j for $j = i_1, i_2, \cdots, i_k$ (where $i_1 < \cdots < i_k$) and $U_j = X_j$ otherwise. Let $x \in B$. For $x_{i_1}, x_{i_2}, \cdots, x_{i_k}$, there exist ϵ_{i_k} such that $U_{i_k} = B_{\epsilon_{i_k}}(x_{i_k}) \subseteq U_{i_k}$.

Choose $\epsilon = \min\left\{\frac{\epsilon_{i_1}}{2^{i_1}}, \cdots, \frac{\epsilon_{i_k}}{2^{i_k}}\right\}$ and let $y \in B_\epsilon(x)$. Since $d(x,y) = \sum_{j=1}^{\infty} \frac{\overline{d_j(x_j,y_j)}}{2^j} < \epsilon$ and, for any $j = 1, \cdots, k$, $\frac{\overline{d_j(x_{i_j},y_{i_j})}}{2^j} < \sum_{j=1}^{\infty} \frac{\overline{d_j(x_{i_j},y_{i_j})}}{2^j}$, we have that $\frac{\overline{d_j(x_{i_j},y_{i_j})}}{2^j} < \epsilon$. $\epsilon = \min\left\{\frac{\epsilon_{i_j}}{2^j}\right\}_{j=1}^{i_k}$, for each $j = 1, \cdots, k$, it is true that $\frac{\overline{d_j(x_{i_j},y_{i_j})}}{2^j} < \frac{\epsilon_{i_j}}{2^j}$. Thus, $d_j(x_{i_j}, y_{i_j}) < \epsilon_{i_j}$. For all j, $y_{i_j} \in B_{\epsilon_{i_j}} \subseteq U_{i_j}$, so $y \in B$.

Now, let $B_\epsilon(x)$ be a basic open set in $\varprojlim(X_i, f_i)$. Choose $k \in \mathbb{N}$ such that $\frac{1}{2^k} < \frac{\epsilon}{2}$ and ϵ_i such that $\sum_i^k \frac{\epsilon_i}{2^i} < \frac{\epsilon}{2}$. Let $B = U_1 \times \cdots \times U_k \times X_{k+1} \times \cdots$, where $U_i = B_{\epsilon_i}(x_i)$, which is open in the product topology. Now, let $y \in B$. It will be shown that $y \in B_\epsilon(x)$.

$$\begin{aligned}
d(x,y) &= \sum_{i=1}^{\infty} \frac{\overline{d_i(x_i,y_i)}}{2^i} \\
&= \sum_{i=1}^{k} \frac{\overline{d_i(x_i,y_i)}}{2^i} + \sum_{i=k+1}^{\infty} \frac{\overline{d_i(x_i,y_i)}}{2^i} \\
&\leq \sum_{i=1}^{k} \frac{\overline{d_i(x_i,y_i)}}{2^i} + \sum_{i=k+1}^{\infty} \frac{1}{2^i} \\
&= \sum_{i=1}^{k} \frac{\overline{d_i(x_i,y_i)}}{2^i} + \frac{1}{2^k} \\
&< \sum_{i=1}^{k} \frac{\overline{d_i(x_i,y_i)}}{2^i} + \frac{\epsilon}{2} \\
&< \sum_{i=1}^{k} \frac{\epsilon_i}{2^i} + \frac{\epsilon}{2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

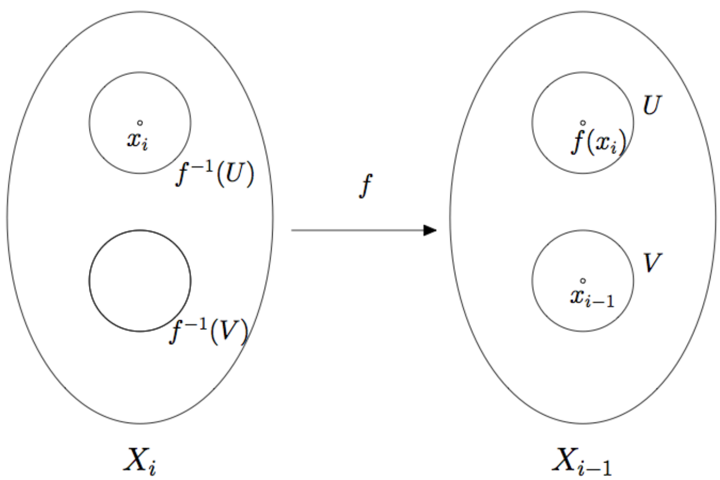
$d(x,y) < \epsilon$, so $y \in B_\epsilon(x)$. In conclusion, since open sets in $\varprojlim(X_i, f_i)$ are also open in the product topology, and vice versa, $\varprojlim(X_i, f_i)$ has the product topology.

■

In the case of compactness, we must add the condition that each X_i is Hausdorff, as well as compact. A topological space, X, is *Hausdorff* if, for each $x, y \in X$, there exist open sets $U \supset x$ and $V \supset y$ such that $U \cap V = \emptyset$.

Proposition 2.3.4. *If* $\{X_i\}_{i=0}^{\infty}$ *is a sequence of compact, Hausdorff spaces, then* $\varprojlim(X_i, f_i)$ *is compact.*

Proof. Let $x \in \prod_{i=1}^{\infty} X_i \setminus X_\infty$, where $x = (x_0, x_1, \cdots)$. Since x is not in the inverse limit space, there exists $i \in \mathbb{N}$ such that $f(x_i) \neq x_{i-1}$. Since X_{i-1} is a Hausdorff space, there exist open sets $U \supset f(x_i)$ and $V \supset x_i$. Consider the sets $f^{-1}(U)$, $f^{-1}(V) \in X_i$.



Since f_i is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are both open. $x_i \in f^{-1}(U)$, so $x_i \notin f^{-1}(V)$ (because $f^{-1}(V)$ maps to a set that is disjoint from U, which contains $f(x_i)$). Now, consider the open set $\mathcal{U} = U \times f^{-1}(V) \times \prod_{j \neq i-1,i} X_i$. For any $y \in \mathcal{U}$, $f(y_i) \neq y_{i-1}$, so $y \notin X_\infty$. $\mathcal{U}$ is an open neighborhood of x that lies entirely in X_∞^c, so X_∞^c is open and X_∞ is closed.

The product space of compact spaces is compact, so X_∞ is a closed subset of a compact space and therefore compact.

■

Then, if each X_i is a continuum, $\underleftarrow{lim}(X_i, f_i)$ is a continuum.

For the next two results, we move our focus from the spaces to the bonding maps.

Proposition 2.3.5. *If $\{X_i, f_i(x)\}_{i=0}^{\infty}$ is a sequence of topological spaces with associated 1-1 and onto maps, then $\underleftarrow{lim}(X_i, f_i)$ is homeomorphic to X_0.*

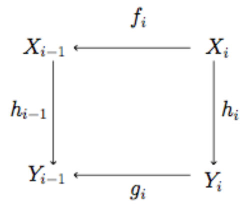
Proof. Let each f_i be 1-1 and onto and define $f : X_0 \to X_\infty$ by $f(x_0) = \pi_0^{-1}(x_0)$. Since each f_i is onto, every $x_0 \in X_0$ maps to an element of X_∞ and since each f_i is 1-1, that element is unique. Then, f is well-defined. If, for some $x, y \in X_\infty$, $x = y$, then $x_0 = y_0$, so f is 1-1. For any $x \in X_\infty$ with first coordinate x_0, $f(x_0) = x$, so f is onto. The projection map is continuous and open, so f is bicontinuous. Then, f is a homeomorphism from X_∞ to X_0. ■

Lemma 2.3.6. *([Mun84])*

Let $f : A \to \prod_{\alpha \in J} X_\alpha$ be given by the equation $f(a) = (f_\alpha(a))_{\alpha \in J}$, where $f_\alpha : A \to X_\alpha$ for each α. Let $\prod X_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

Proposition 2.3.7. *Let $h_i : X_i \to Y_i$ be homeomorphisms, for all i, where*
$X_\infty = \underleftarrow{lim}(X_i, f_i)$ and $Y_\infty = \underleftarrow{lim}(Y_i, g_i(x))$.

Let the following commute for all $i \geq 1$



Then, $h = (h_1, h_2, \cdots) : X_\infty \to Y_\infty$ is a homeomorphism.

Proof. Let h_i be homeomorphisms as described above, and $h = (h_1, h_2, \cdots)$. It will be shown that h is a homeomorphism. To prove that h is one to one, assume $h(x) = h(\tilde{x})$ for some $x, \tilde{x} \in X_\infty$. Then, $(h_1(x_1), h_2(x_2), \cdots) = (h_1(\tilde{x_1}), h_2(\tilde{x_2}), \cdots)$ and $h_i(x_i) = h_i(\tilde{x_i})$ for all i. Since each h_i is one to one, $x_i = \tilde{x}_i$ and $x = \tilde{x}$.

Now, let $y = (y_1, y_2, \cdots) \in Y_\infty$. Each h_i is onto, so for all i there exist x_i such that $h_i(x_i) = y_i$. If we define $x = (x_1, x_2, \cdots)$, then $h(x) = y$. It remains to be shown that $x \in X_\infty$.

$$\begin{aligned} x_{i-1} &= h_{i-1}^{-1}(y_{i-1}) \\ &= h_{i-1}^{-1}(g_i(y_i)) \\ &= h_{i-1}^{-1}(g_i(h_i(x_i))) \\ &= f_i(x_i) \end{aligned}$$

Thus, $x \in X_\infty$ and h is onto.

To prove that h is continuous, we can apply the previous lemma, where $A = X_\infty$

and $J = \mathbb{N}$. It has been previously shown that the product topology is equivalent to the topology in the inverse limit space, so the lemma applies. Since each h_i is continuous, h is continuous. h^{-1} satisfies the same properties that were applied in the continuity argument, so h^{-1} is also continuous.

■

Chapter 3

Smale Spaces

In [Sma66], Smale defines a class of dynamical systems known as Axiom A. Ruelle adapted this idea in [Rue04] to come up with the definition of a Smale space, the basic idea being that in the space there is a contracting direction and an expanding direction, and that locally, they intersect at exactly one point.

3.1 Definitions

Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism of X. A point $x \in X$ is *non-wandering* if, for every non-empty open set, U, containing x, there is a positive integer n such that $U \cap f^n(U)$ is non-empty. $\Omega(f)$ is the set of all such points, called the *non-wandering set.* Also, x is called a *periodic point* if $f^n(x) = x$ for some positive integer n. If $f : M \to M$ is a diffeomorphism, then a closed invariant subset $\Lambda \subseteq M$ is *hyperbolic* if the tangent bundle $T(M)$ restricted to Λ splits as a direct sum, $T(M)|\Lambda = E^u \oplus E^s$, invariant under the derivative Df of f and such that $Df|E^u$ is an expansion and $Df|E^s$ is a contraction. That is, there exist constants $A, B > 0$ and $\mu > 1$ such that for all $n \in \mathbb{N}$, $v \in E^u$, and $w \in E^s$, we have $|Df^n(v)| \geq A\mu^n|v|$ and $|Df^n(w)| \leq B\mu^{-n}|w|$. In simpler terms, in a hyperbolic dynamical system, the tangent space has 2 parts: where the derivative of the map contracts and where it expands. An *attractor* is an invariant set that attracts open sets of points. An attractor is *expanding* if it has the same dimension as E^u, that is the part with the expanding derivative.

Smale defined an Axiom A system to be a compact manifold, M, together with a diffeomorphism $f : M \to M$ whose non-wandering set is both hyperbolic and compact, and if the set of periodic points of f is dense in $\Omega(f)$.

Now, we will define a Smale space.

Definition 3.1.1. *Let (X, d) be a compact metric space and $f : X \to X$ be a homeomorphism. The triple (X, d, f) is a Smale space if there exist constants $\epsilon_X > 0$ and $0 < \lambda < 1$, as well as a mapping*

$$[\cdot, \cdot] : \{(x, y) \in X \times X | d(x, y) \le \epsilon_X\} \mapsto [x, y] \in X$$

satisfying properties $(S1)$ through $(S7)$ below. For $x \in X$ and $0 < \epsilon \le \epsilon_X$, we denote

$$X^s(x, \epsilon) = \{y \,|\, [x, y] = y,\ d(x, y) \le \epsilon\}$$
$$X^u(x, \epsilon) = \{y \,|\, [y, x] = y,\ d(x, y) \le \epsilon\}$$

these are called the local stable and unstable sets of x.

(S1) $[\cdot, \cdot]$ *is continuous*

(S2) $[x, x] = x$ *for all* $x \in X$

(S3) $[[x, y], z] = [x, z]$ *whenever both sides are defined*

(S4) $[x, [y, z]] = [x, z]$ *whenever both sides are defined*

(S5) $f([x, y]) = [f(x), f(y)]$ *whenever both sides are defined*

(S6) $d(f(y), f(z)) \leq \lambda d(y, z)$ *if* $y, z \in X^s(x, \epsilon_X)$

(S7) $d(f^{-1}(y), f^{-1}(z)) \leq \lambda d(y, z)$ *if* $y, z \in X^u(x, \epsilon_X)$

Alternatively, the local stable and unstable sets can be defined below. They are shown to be equivalent in Propositions 3.3.6 and 3.3.7.

$X^s(x, \epsilon) = \{y \in X \mid d(f^n(x), f^n(y)) \leq \epsilon \; \forall n \geq 0\}$

$X^u(x, \epsilon) = \{y \in X \mid d(f^{-n}(x), f^{-n}(y)) \leq \epsilon \; \forall n \geq 0\}$

A space is *totally disconnected* if its only connected subspaces are one-point sets. Smale spaces with totally disconnected stable sets are known to be shifts of finite type, and shifts of finite type are known to be inverse limits of one-sided shifts of finite type. This relationship provided some of the motivation to Wieler's work. Putnam also provided some properties of these spaces ([Put12]).

In addition, an *irreducible* Smale space has the added condition that it is non-wandering and has a forward orbit that is dense in the space.

3.2 Examples

Example 3.2.1. *To see an example of a Smale space with totally disconnected local stable sets, we can look back at the idea of Williams' solenoid. In [Wil74], he showed that each point of an* n*-solenoid has a neighborhood of the form (Cantor set)* $\times$ *(*n*-disk). The Cantor set is totally disconnected and represents the contracting direction (the local stable set) and the* n*-disk presents the expanding direction (the local unstable set).*

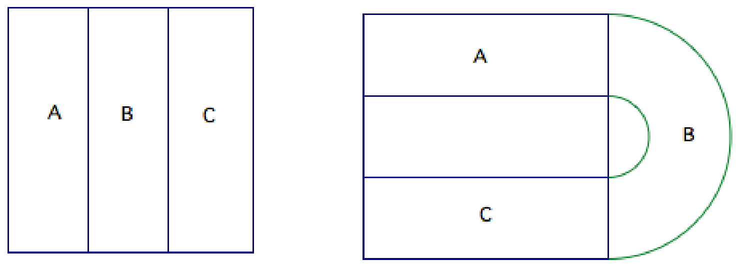


Figure 3.1: One iteration of Horseshoe Map.

The previous example demonstrated that an inverse limit space could be a Smale space. Next is an example that shows this is not always the case.

Example 3.2.2. *In Example 2.2.2, the Topologist's Sine Curve was homeomorphic to the inverse limit space of unimodal bonding maps. The maps can be amended so that the orbit of the critical point is dense, which means that there will be 'hooks' appearing densely in the resulting inverse limit space. The consequence of these hooks is that the space is not only not hyperbolic, it is nowhere hyperbolic. See [BD99] for the explicit construction.*

Example 3.2.3. *Another widely known and examined example is Smale's Horseshoe. It is defined on a rectangle. The domain is then squished in the vertical direction, then stretched in the horizontal direction, and finally folded over.*

The set that remains invariant under one iteration of both forward and backward maps are the four corner regions, shown in Figure refhorse and the set that remains invariant under two iterations of both forward and backward maps are the four corner regions of the four corner regions, shown in Figure refcorners.

Then, the invariant set, which in dynamical systems is referred to as the attractor,

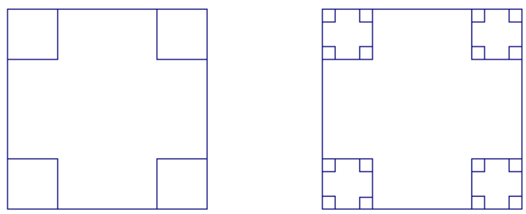


Figure 3.2: Invariant set of one iteration in both forward and backward direction.

has the structure of the Cantor set, meaning it is uncountably infinite and has zero measure. The complete transversal (defined in Chapter 4) is then zero-dimensional.

The horseshoe is an example of an Axiom A system. One of the main properties of an Axiom A system is that its non-wandering set has hyperbolic structure, and is the closure of its periodic points. Smale showed that there is only hyperbolic structure where there is recurrence, i.e., a wandering set ([Wil67]). Then, since the non-wandering set is not the entire manifold for the Smale's Horseshoe, it is not a Smale space. However, we can restrict to the map to just the non-wandering set, and then it is a Smale space.

It was mentioned before that Smale spaces with totally disconnected stable sets are known to be shifts of finite type. This is very useful for examining the dynamics of the horseshoe map, which is explained very well by Shub ([Shu05]). First, we will define the horseshoe slightly differently.

The horseshoe is defined as $\Lambda = \{z : f^n(z) \in A \cup C \ \forall n \in \mathbb{Z}\}$. *Next, we define* Σ *to be the set of bi-infinite sequence* $\mathbf{a} = (a_n)$ *consisting of 0 and 1's and* $\sigma : \Sigma \to \Sigma$ *by* $\sigma(\mathbf{a}) = (a_{n+1})$. *This is known as the (right) shift map, which is a homeomorphism. The dynamics of* Λ *can be thought of as being coded into sequences by* σ. *Given* $\mathbf{a} \in \Sigma$, *there is a unique $z \in \Lambda$ such that $f^n(z) \in A$ whenever $a_n = 1$ and $f^n(z) \in C$ whenever $a_n = 0$. The last thing of note is that all shifts of finite type are the inverse limit of a one-sided shift of finite type.*

3.3 Basic Results

The first two results in this section are stated in [Put12] by Putnam, with the details of the proofs filled in here, and illustrate two important properties of Smale spaces.

Lemma 3.3.1. *$d_2 : X \times X \to \mathbb{R}$ defined by $d_2((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}$ is a metric.*

Proof. Let d_2 be defined on $X \to X$ where d is a metric on X. Clearly d is non-negative and symmetric.

$$\begin{aligned} d_2((x,y),(u,v)) = 0 &\Leftrightarrow \max\{d(x,u), d(y,v)\} = 0 \\ &\Leftrightarrow d(x,u) = 0 \text{ and } d(y,v) = 0 \\ &\Leftrightarrow x = u \text{ and } y = v \\ &\Leftrightarrow (x,y) = (u,v) \end{aligned}$$

To prove the triangle inequality, let $(a,b) \in X \times X$. If $d(x,u) \leq d(y,v)$, then

$$\begin{aligned} d_2((x,y),(u,v)) &= \max\{(x,u),(y,v)\} \\ &= d(y,v) \\ &\leq d(y,b) + d(b,v) \\ &\leq \max\{d(x,a), d(y,b)\} + \max\{d(a,u), d(b,v)\} \\ &= d_2((x,y),(a,b)) + d_2((a,b),(u,v)) \end{aligned}$$

Otherwise, $d(x,u) < d(y,v)$, in which case

$$\begin{aligned} d_2((x,y),(u,v)) &= \max\{(x,u),(y,v)\} \\ &= d(y,v) \\ &\leq d(y,b) + d(b,v) \\ &\leq \max\{d(x,a), d(y,b)\} + \max\{d(a,u), d(b,v)\} \\ &= d_2((x,y),(a,b)) + d_2((a,b),(u,v)) \end{aligned}$$

In either case, the triangle inequality holds, so d_2 is a metric.

■

The next result shows that the local stable and unstable sets intersect at exactly one point.

Proposition 3.3.2. *For ϵ sufficiently small enough, $d(x,y) \leq \epsilon$ implies*

$$X^s(x,\epsilon_X) \cap X^u(y,\epsilon_X) = \{[x,y]\}$$

Proof. Let (X,d,f) be a Smale space as defined above. We will show there exists $0 < \epsilon \leq \epsilon_X$ such that $d(x,y) < \epsilon$ implies both that $[x,y] \in X^s(x,\epsilon_X) \cap X^u(y,\epsilon_X)$ and that $z \in X^s(x,\epsilon_X) \cap X^u(y,\epsilon_X)$ implies $z = [x,y]$.

First note that $[,]$ is uniformly continuous because it is continuous on a compact set, $X \times X$. This means there is a $\delta > 0$ such that $d_2((x,x),(x,y)) < \delta$ implies $d([x,x],[x,y]) < \epsilon_X$. Therefore, we choose $\epsilon = \min\{\delta, \epsilon_X\}$, and let $d(x,y) \leq \epsilon$. Then since $[x,x] = x$, we have that $\epsilon > d(x,y) = d_2((x,x),(x,y))$ implies $d([x,x],[x,y]) \leq \epsilon_X$, and so $d(x,[x,y]) \leq \epsilon_X$. Also, by $(S4)$, $[x,[x,y]] = [x,y]$, so $[x,y] \in X^s(x,\epsilon_X)$.

$X^u(y, \epsilon_X) = \{x \,|\, [x, y] = x,\ d(x, y) \leq \epsilon_X\}$. By $(S3)$ and $(S2)$, $[[x, y], x] = [x, x] = x$, and we have already shown that $d([x, y], x) \leq \epsilon_X$, so $[x, y] \in X^u(y, \epsilon_X)$ and $[x, y] \in X^s(x, \epsilon_X) \cap X^u(y, \epsilon_X)$.

Now it remains to be shown that if $z \in X^s(x, \epsilon_X) \cap X^u(y, \epsilon_X)$, then $z = [x, y]$. $z \in X^s(x, \epsilon_X)$ implies $[x, z] = z$. $z \in X^u(y, \epsilon_X)$ implies $[z, y] = z$. Therefore, $z = [z, z] = [[x, z], [z, y]] = [x, [z, y]] = [x, y]$ and so $X^s(x, \epsilon_X) \cap X^u(y, \epsilon_X) = \{[x, y]\}$.

■

Proposition 3.3.3. *There exists $0 \leq \epsilon'_X \leq \frac{\epsilon_X}{2}$ such that for any $0 < \epsilon \leq \epsilon'_X$, $X^u(x, \epsilon) \times X^s(x, \epsilon)$ is homeomorphic to its image, which is a neighborhood of x.*

Proof. Let (X, d, f) be a Smale Space and let $x \in X$. We will show that the bracket map, restricted to the domain $X^u(x, \epsilon) \times X^s(x, \epsilon)$, where $\epsilon < \frac{\epsilon_X}{2}$, is a homeomorphism from the domain to a neighborhood of x. If $y \in X^u(x, \epsilon)$ and $z \in X^s(x, \epsilon)$, then $d(x, y) \leq \epsilon$ and $d(x, z) \leq \epsilon$, so by the triangle inequality, $d(y, z) \leq d(x, y) + d(x, z) < \epsilon_X$. This means (y, z) is in the domain of the bracket map, as defined in Defn 3.1.1, so the restricted map is well-defined.

$[,]$ is uniformly continuous, so we can find $0 < \delta \leq \epsilon_X$ such that if $x, y \in X$ and $d(x, y) \leq \delta$, we have $d(x, [x, y]) \leq \frac{\epsilon_X}{2}$ and $d(x, [y, x]) \leq \frac{\epsilon_X}{2}$. Also, we can choose $0 < \epsilon'_x \leq \frac{\epsilon_X}{2}$ such that, for all $y, z \in B_{\epsilon'_X}(x)$, $d(x, [y, z]) \leq \delta$. Note that $\epsilon'_X \leq \epsilon_X$ ensures $d(y, z) \leq \epsilon_X$, which means $[y, z]$ is defined.

Now, define $h(y) = ([y, x], [x, y])$ on the range of $[,]$.Our choice of ϵ'_X means that, for any $\epsilon \leq \epsilon'_X$, $(y, z) \in X^u(x, \epsilon) \times X^s(x, \epsilon)$ guarantees $d(x, [y, z]) \leq \delta$. Any element of the range can be expressed as $[z_1, z_2]$ for some $(z_1, z_2) \in X^u(x, \epsilon) \times X^s(x, \epsilon)$,

$$h([z_1, z_2]) = ([z_1, z_2], x], [x, [z_1, z_2]]) = ([z_1, x], [x, z_2]) = (z_1, z_2)$$

and so $([,] \circ h)([z_1, z_2]) = (z_1, z_2)$, and so h is the inverse of $[,]$. h is clearly continuous, so $X^u(x, \epsilon) \times X^s(x, \epsilon)$ is homeomorphic to its image. ∎

The next two results are from Ruelle ([Rue04]), with details of the proof filled in. They show there are equivalent definitions of the local stable and unstable sets, which make more sense intuitively.

Proposition 3.3.4. *For $\epsilon > 0$ and $x \in X$, $X^s(x, \epsilon) = \{y \in X \mid d(f^n(x), f^n(y)) \leq \epsilon \; \forall n \geq 0\}$.*

Proof. Let $\epsilon > 0$ and $x \in X$. Suppose $y \in X^s(x, \epsilon)$, given by Definition 3.1.1. By an inductive version of $(S6)$, $d(f^n(x), f^n(y)) \leq \lambda^n d(x, y) < \lambda^n \epsilon < \epsilon$, for all $n \geq 0$. The other direction requires a little more work.

Since [,] is uniformly continuous, we can choose $0 < \delta < \epsilon$ such that $d(x, y) < \delta$ implies $d(x, [y, x]) < \epsilon$. Suppose $d(f^n(x), f^n(y)) < \delta$ for all $n \geq 0$. Then, $d(f^n(x), [f^n(y), f^n(x)]) < \epsilon$, which is equivalent to $d(f^n(x), f^n[y, x]) < \epsilon$, by $(S5)$.

$$[f^n[y, x], f^n(x)] = [[f^n(y), f^n(x)], f^n(x)] = [f^n(y), f^n(x)] = f^n[y, x]$$

This means that $f^n[y, x] \in X^u(f^n(x), \epsilon)$. By an inductive version of $(S7)$, $d([y, x], x) \leq \lambda^n d(f^n[y, x], f^n(x)) \leq \lambda^n \epsilon$. Since this is true for all $n \geq 0$, $d(x, [x, y]) = 0$. Using this fact,

$$[x, y] = [x, [x, y]] = [[y, x], [x, y]] = y$$

Thus, $y \in X^s(x, \delta) \subset X^s(x, \epsilon)$, and the two definitions are equivalent.

∎

Proposition 3.3.5. *For* $\epsilon > 0$*, and* $x \in X$*,* $X^u(x, \epsilon) = \{y \in X \mid d(f^{-n}(x), f^{-n}(y)) \leq \epsilon \ \forall n \geq 0\}$

Proof. Let $\epsilon > 0$ and let $x \in X$. Suppose $y \in X^u(x, \epsilon)$, given by definition 3.1.1. By an inductive version of $(S7)$, $d(f^{-n}(y), f^{-n}(x)) \leq \lambda^n d(x, y) < \lambda^n \epsilon < \epsilon$, for all $n \geq 0$. To show containment in the other direction, choose $0 < \delta < \epsilon$ such that $d(x, y) < \delta$ implies $d(x, [x, y]) < \epsilon$ and suppose $d(f^{-n}(x), f^{-n}(y)) < \delta$, for all $n \geq 0$. Then, $d(f^{-n}(x), [f^{-n}(x), f^{-n}(y)]) < \epsilon$, which is equivalent to $d(f^{-n}(x), f^{-n}[x, y]) < \epsilon$, by an inductive version of $(S5)$. $[f^{-n}(x), f^{-n}[x, y]] = [f^{-n}(x), f^{-n}(y)] = f^{-n}[x, y]$, so $f^{-n}[x, y] \in X^s(f^{-n}(x), \epsilon)$. By an inductive version of $(S6)$, $d(x, [x, y]) \leq \lambda^n d(f^{-n}(x), f^{-n}[x, y]) \leq \lambda^n \epsilon$. This holds for all $n \geq 0$, so $[x, y] = x$. This implies $[y, x] = [y, [x, y]] = y$, so $y \in X^u(x, \delta) \subset X^u(x, \epsilon)$, and the two definitions are equivalent.

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Chapter 4

Laminations

Now, we move from dynamical systems to the area of manifolds. The idea is to decompose a manifold into 'parallel' submanifolds of smaller dimension. This will require a plethora of new definitions.

4.1 Definitions

Let M be an n-manifold and a locally compact Polish space, that is a separable, metrizeable space. If $x \in M$, then $C_x(M)$ is the set of all C^∞ curves $\alpha : (-\epsilon, \epsilon) \to M$, where $\epsilon > 0$ and $\alpha(0) = x$. We will define an equivalence relation on $C_x(M)$ such that $\alpha \sim \beta$ if, for any local chart (U, ϕ), $x \in U$, we have $(\frac{d(\phi\circ\alpha)}{dt}(t)|_{t=0} = (\frac{d(\phi\circ\beta)}{dt})|_{t=0}$. Then the quotient $C_x(M)/\sim$ is called the *tangent space to M at x*, denoted $T_x(M)$.

In the Euclidean space $\mathbb{F}^n$, a *rectangular neighborhood* is an open subset $B = J_1 \times \cdots \times J_n$ where each J_i is an open interval in the ith coordinate. A pair (U, ϕ) is a *foliated chart of codimension q* if U is an open set and ϕ is a diffeomorphism that maps U to $B_\tau \times B_\pitchfork$, where B_τ is a rectangular neighborhood in $\mathbb{F}^q$ and $B_\pitchfork$ is a rectangular neighborhood in $\mathbb{F}^{n-q}$. The open sets are called *flow boxes.* A *plaque* of the foliated chart is defined as $P_y = \phi^{-1}(B_\tau \times \{y\})$, where $y \in B_\pitchfork$, and a *local transversal* if defined as $\phi_i^{-1}(\{x\} \times B_\pitchfork)$. An *atlas of dimension q* is a collection of

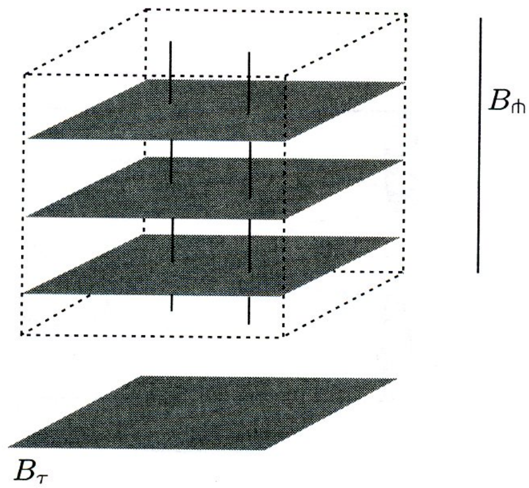


Figure 4.1: Foliation of a 3-dimensional manifold. From [CC00].

local charts that cover M with the property that, if $(U, \phi), (U', \phi') \in \mathcal{A}$, where U, U' are not disjoint, then $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ is a C^r diffeomorphism. The map $\phi' \circ \phi^{-1}$ can be thought of as a change of coordinates. Figure reffoliation shows a a foliation of a 3-dimensional manifold.

If $r \geq 1$, and M, N are manifolds, then a C^r map $f : M \to N$ is an *immersion* if, for every $x \in M$, $Df(x) : T_xM \to T_yN$, $y = f(x)$, is injective. Let $\mathcal{F} = \{L_\lambda\}_{\lambda \in \mathcal{L}}$ be a decomposition of M into connected, immersed sub manifolds of dimension $k = n - q$. Also, let $\{U_\alpha, \phi_\alpha\}_{\alpha \in \mathcal{A}}$ be an atlas of foliated charts of codimension q with the added assumption that, for each $\alpha \in \mathcal{A}$ and $\lambda \in \mathcal{L}$, $L_\lambda \cap U_\alpha$ is a union of plaques. Then, $\mathcal{F}$ is a *foliation of M of codimension q and dimension k*. Each L_λ is a *leaf* of the foliation and $(M, \mathcal{F})$ is called a *foliated manifold.*

A *lamination* of class C^r is a base space M endowed with an equivalence class $\mathcal{L}$ of foliated atlas, where two foliated atlases of class C^r are *equivalent* if their union is a foliated atlas. The leaves L of a lamination are the smallest path connected sets such that if L meets a plaque P, the $P \subset L$. The *complete transversal* is a transversal meeting all leaves, and the *codimension* of the lamination $\mathcal{L}$ is the dimension of the complete transversal associated to any atlas. Conceptually what this means is that a lamination is like a foliation, but it is only over a subset of the manifold, as if there were gaps between the lamination. In certain situations, it is possible to close these gaps and make the lamination into a foliation.

A topological space is zero-dimensional at a point p, if, for all $\epsilon > 0$, there exists a clopen set that is a subset of the open ball $B_\epsilon(p)$ and contains p. Rojo's results are for codimension zero laminations, which means the complete transversal has zero dimension. Also of note is the relationship between totally disconnected and zero dimensional, which is shown in Section 4.3.

4.2 Examples

In section 5.2, we will see that under the right conditions, a lamination can found by constructing a quotient space. This following is an example of this.

Example 4.2.1. *Let* $C = \{0,1\}^{\mathbb{Z}}$ *and* $\sigma((x_i))_k = x_{k+1}$ *be a shift map for* C*.* $\mathbb{R} \times C$ *has a horizontal foliation, where the leaves are given by* $\mathbb{R} \times \{x\}$ *for* $x \in C$*. Now let the action of* $\mathbb{Z}$ *on* $\mathbb{R} \times C$ *be defined by* $(\lambda, x) + l = (\lambda - l, \sigma^l(x))$*. The foliation on* $\mathbb{R} \times C$ *is invariant under the action of* $\mathbb{Z}$*, so it induces a lamination of class* C^∞ *on the quotient space* $(\mathbb{R} \times C)/\mathbb{Z}$*.*

4.3 Basic Results

In a compact, or locally compact metric space, totally disconnected and zero dimensional are equivalent. A zero dimensional metric space is easily shown to be totally disconnected. Proving a compact, (or locally compact) totally disconnected set is zero dimensional will require more effort.

Proposition 4.3.1. *Let M be a metric space. If M is zero dimensional, then it is totally disconnected.*

Proof. Suppose M is zero dimensional and it is not totally disconnected, that is there exists a connected set $E \subset M$ that contains two distinct points, p and q. Let $0 < \epsilon < d(p, q)$. Since M is zero dimensional, there exists a clopen set C that is a subset of $B_\epsilon(p)$ and contains p. Note that $C \cap E$ is clopen. In a connected space, the only clopen sets are the empty set and the entire space. But $C \cap E$ contains p but does not contain q, so it is neither. This implies $C \cap E$, and therefore E is not connected. ∎

Proposition 4.3.2. *If M is compact and totally disconnected, then it is zero dimensional.*

Proof. First, define $\mathcal{C}_n = \left\{B_{\frac{1}{n}}(x) : x \in M\right\}$ be a collection of open balls. Since M is compact, there exists a finite subset of $\mathcal{C}_n$ that covers M. For any such subset S, let $D_n(S)$ denote the minimum number of balls needed to cover the space. Notice $D_n(S) \leq D_n(M)$.

Let $p \in M$. We will now define (C_n) to be a recursive sequence of sets satisfying the following properties:

(a) C_n is clopen in M and $p \in C_n$.

(b) $C_n \subset C_{n-1}$ if $n > 0$

(c) For $n > 0$, $D_n(C_n)$ is the minimum value of $D_n(S)$ as S varies over sets which satisfy

$$S \subset C_{n-1},\ S \text{ is clopen, and } p \in S \qquad (\star)$$

Let $C_0 = M$, and assume C_{n-1} has been defined, for some $n > 0$. Then, C_{n-1} is clopen in M and so $S = C_{n-1}$ satisfies $(\star)$. There is at least one set that satisfies $(\star)$, so we can choose C_n to be one that minimizes $D_n(S)$.

Now define $C = \cap_{n \geq 0} C_n$. Clearly, $p \in C$, and in fact we will show $C = \{p\}$. To do this, suppose C contains more than one point. C is compact, since each C_n is closed, and $p \in C$ so we can write $C = A \cup B$, where A and B are closed, non-empty, disjoint sets, and label the sets so that $p \in A$. A and B are compact, since they are closed in C. If we let $\alpha = \inf\{d(a, b)\ : a \in A,\ b \in B\}$, then we can define $U = \cup\left\{B_{\frac{\alpha}{4}}(a)\ : a \in A\right\}$ and $V = \cup\left\{B_{\frac{\alpha}{4}}(b)\ :\ b \in B\right\}$. $U \cup V$ is open and contains C.

Since (C_n) is a compact, nested sequence, we can find N such that $n \geq N$ implies $C_n \subset U \cap V$. Choose $n \geq N$ such that $\frac{1}{n} < \frac{\alpha}{4}$. Then, any $\frac{1}{n}$ ball that intersects V does not intersect U.

Using our notation, $D_n(C_n)$ is the minimum number of balls of radius $\frac{1}{n}$ needed to cover C_n, so $D_n(C_n \cap U) \leq D_n(C_n)$. $C_n \cap V$ is non-empty, so any cover of C_n includes a ball that intersects $C_n \cap V$, and that ball must be disjoint from U. $U \supset C_n \cap U$, so that ball is not in a minimal cover of $C_n \cap U$, and so $D_n(C_n \cap U) < D_n(C_n)$. C_n is clopen and U is open, so $C_n \cap U$ is open. Also, $M \setminus V$ is closed, so $C_n \cap U = C_n \cap (M \setminus V)$ is closed. Therefore S is clopen. Additionally, $p \in C_n \cap A$, so $p \in C_n \cap U$. Then, $C_n \cap U$ satisfies ($\star$), which is a contradiction because of how we chose C_n, and so $C = \{p\}$.

Finally, to show M is zero dimensional at p, notice $C \subset B_\epsilon(p)$ for any $\epsilon > 0$. (C_n) is a nested, compact sequence, so there exists N such that $C_N \subset B_\epsilon(p)$, which means $p \in B_\epsilon(p)$. C_N is also clopen, by definition, so M is zero dimensional at p. $p \in M$ was chosen arbitrarily, so M is zero dimensional. ■

This result can be strengthened by only needing M to be locally compact, which means that for every $p \in M$ there is an $\epsilon > 0$ such that $\overline{B_\epsilon(p)}$ is compact. Local compactness is a stronger statement than compactness.

Proposition 4.3.3. *If M is locally compact and totally disconnected, then it is zero dimensional at every point $p \in M$.*

Proof. Let $\epsilon > 0$ and $p \in M$. For any $0 < \gamma < \epsilon$, $\overline{B_\gamma(p)}$ is a closet subset of $\overline{B_\epsilon(p)}$, which is a compact metric space, so $\overline{B_\gamma(p)}$ is compact. It is compact and totally disconnected, so by the previous proposition, there is a subset $S \subset \overline{B_\gamma(p)}$ that contains p that is clopen in $\overline{B_\gamma(p)}$. A is closed in $\overline{b_\gamma(p)}$ which is closed in M, so S is closed in M. Both $B_\gamma(p)$ and A are open in $\overline{B_\gamma(p)}$, so $A = B_\gamma(p) \cap A$ is open in $B_\gamma(p)$. Then, since $B_\gamma(p)$ is open in M, so is A. Thus, A is clopen in M, and M is zero dimensional at p. $p \in M$ was chosen arbitrarily, so M is zero dimensional. ■

Chapter 5

Connections

So far, we have discussed three different, but related topics: inverse limits, Smale spaces and laminations. We will now relate these topics further by looking at results by Wieler and Rojo, the former relating to Smale spaces and the latter relating to laminations and both relating to inverse limit spaces. Both built on the work of R.F. Williams ([Wil74]). We will look at each of the results separately, with some examples and then discuss how they are related.

5.1 Inverse Limits and Smale Spaces

Williams took previous results characterizing attractors with hyperbolic structure and added the condition that the attractors be expanding. With this condition he was able to provide complete proofs of results relating any expanding hyperbolic attractor to an inverse limit space. He associated an expanding attractor, which is a topological idea, with the concept of a bonding map being expanding, which is a dynamical idea. Wieler's goal was to generalize these result for Smale spaces. First, we will state the theorems noting that $\hat{Y} = \varprojlim(Y, g)$, $\hat{g}$ is the associated mapping and $\hat{d}$ is the associated metric, both simply the restrictions of g and d to $\hat{Y}$. Also, this version of the theorems comes from [Wie12b] , which differs slightly from [Wie12a].

Let (Y, d) be a compact metric space, and let $g : Y \to Y$ be continuous and surjective. We will say that (Y, d, g) satisfies Axiom 1 and 2 if there exist constants $\beta > 0$, $K \geq 1$, and $0 < \gamma < 1$ such that

Axiom 1: *if* $d(x, y) \leq \beta$ *then*

$$d(g^K(x), g^K(y)) \leq \gamma^K d(g^{2K}(x), g^{2K}(y)),$$

and

Axiom 2: *for all* $x \in Y$ *and* $0 < \epsilon \leq \beta$,

$$g^K(B(g^K(x), \epsilon)) \subseteq g^{2K}(B(x, \gamma\epsilon)).$$

Theorem A. *If* (Y, d, g) *satisfies Axioms 1 and 2 then* $(\hat{Y}, \hat{d}, \hat{g})$ *is a Smale space with totally disconnected local stable sets. Moreover,* $(\hat{Y}, \hat{d}, \hat{g})$ *is an irreducible Smale space if and only if* (Y, d, g) *is non-wandering and has a dense forward orbit.*

Theorem B. *Let* (X, d, f) *be an irreducible Smale space with totally disconnected local stable sets. Then* (X, d, f) *is topologically conjugate to an inverse limit space* $(\hat{Y}, \hat{\delta}, \hat{\alpha})$ *such that* (Y, δ, α) *satisfies Axiom 1 and 2.*

Note that two function, f and g are topologically conjugate if there exists a homeomorphism h such that $h \circ f = g \circ h$. The first axiom is a weakened version of the condition that g be locally expanding and the second axiom is a weakened version of a condition that g be locally open. Smale spaces have a contracting and an expanding direction, so if the inverse limit is a Smale space, we need to guarantee those directions. The inverse limit has a natural contracting direction, and this axiom guarantees the expanding direction. These two directions ensure the hyperbolic structure of the Smale space. This axiom also implies g is finite-to-one. What follows are two counterexamples that show the necessity of both Axioms in the theorem.

Example 5.1.1. *Example 3.2.2 was not a Smale space because of a lack of hyperbolic structure. It fails Axiom 1 because it is not expanding near the critical point.*

Next is an example that fails Axiom 2 and is not a Smale space.

Example 5.1.2. *In [Wie12b], Wieler suggests the following counterexample for Axiom 2. Let* $\Sigma^+_{\{0,1\}}$ *and* $\Sigma^+_{\{0,2\}}$ *be the full one-sided shifts on* $\{0,1\}$ *and* $\{0,2\}$ *respectively, and use the metric* $d(\mathbf{x},\mathbf{y}) = 2^{-\min\{n|x_n\neq y_n\}}$. *Then, let* $Y = \Sigma^+_{\{0,1\}} \cup \Sigma^+_{\{0,2\}}$ *and let* g *be the left shift map.*

Now, let $\beta > 0, K \geq 1, N \geq 2K$ *and* $0 < \gamma < 1$ *and choose* $\mathbf{x},\mathbf{y} \in Y$ *defined by*

$$x_n = \begin{cases} 1 & \text{if } n = N+K \\ 0 & \text{if } n \geq N+K \end{cases} \qquad \text{and} \qquad y_n = \begin{cases} 2 & \text{if } n = N \\ 0 & \text{if } n \neq N \end{cases}$$

Then, $d(g^k(\mathbf{x}),\mathbf{y}) = 2^{-N}$, *so* $\mathbf{y} \in B(g^k(\mathbf{x}), 2^{-N})$. *If* $\mathbf{z} \in B(\mathbf{x}, \gamma 2^{-N})$, *then* $(g^{2K}(z)_{N-2K} = z_N = x_N = 0$. *However,* $(g^{2K}(y)_{N-2K} = y_N = 2$, *so* $g^K(B(g^k(\mathbf{x}), 2^{-N}) \not\subseteq g^{2K}(B(\mathbf{x}, \gamma 2^{-N}))$ *and Axiom 2 fails.*

The inverse limit of the previous space is conjugate to $(\Sigma_{\{0,1\}} \cup \Sigma_{\{0,2\}}, S)$ *where the shifts are now full shifts and* S *is again the left shift map. To see this is not a Smale space, let* $\mathbf{x} \in \Sigma_{\{0,1\}}$ *and* $\mathbf{y} \in \Sigma_{\{0,2\}}$ *be arbitrarily close (that is* N *is arbitrarily large). Using the alternate definitions of the local stable and unstable sets, we see that* $X^s(\mathbf{x},\epsilon) \subset \Sigma^+_{\{0,1\}}$ *and* $X^u(y,\epsilon) \subset \Sigma^+_{\{0,2\}}$. *The intersection of these two sets is empty, which means the inverse limit is not a Smale space.*

These two examples show the importance of the space satisfying Axioms 1 and 2 it to be a Smale space.

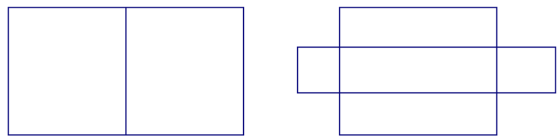
Theorem A is a construction theorem. It constructs a certain type of Smale space if we only have information about (Y, d, g). Both axioms are needed in the proof that $(\hat{Y}, \hat{d}, \hat{g})$ is a Smale space, and Axiom 1 implies g is finite-to-one, which is needed to prove that the local stable sets are totally disconnected. Theorem B is a realization theorem. It says there is a topologically conjugate inverse limit space, but does not construct it. The conditions of the two theorems also imply there is some connection between the system (Y, d, g) being locally open and expanding and the local stable sets of the Smale space being totally disconnected. As mentioned before, Smale spaces with totally disconnected stable sets are known to be inverse limits, so it is not surprising that $(\hat{Y}, \hat{d}, \hat{g})$ is a Smale space under certain conditions.

(Y, d, g) is irreducible if it is non-wandering and has a dense forward orbit. So, if it satisfies Axioms 1 and 2, then (Y, d, g) being irreducible is equivalent to $(\hat{Y}, \hat{d}, \hat{g})$ being irreducible. Thus Theorem A really says that if we begin with (Y, d, g) that satisfies Axioms 1 and 2 as well as being irreducible, then the inverse limit is an irreducible Smale space with totally disconnected local stable sets.

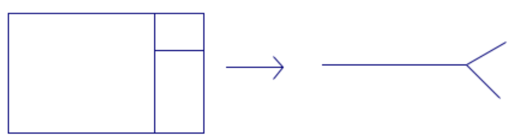
Theorem B starts with an irreducible Smale space with totally disconnected local stable sets. The irreducible condition guarantees there is a Markov partition and the condition on the local stable sets guarantees additional properties of that partition. Both are vital to Wieler's proof. However, although the Smale space must be non-wandering, it is not actually necessary that the space have a dense forward orbit for the theorem to hold.

In the proof of Theorem B, Wieler started with an irreducible space and placed a

Markov partition on it.



This is a partition of irreducible rectangles that either border each other or pass completely through each other, as shown in the figure above. She then created a quotient space using an equivalence relation, which collapses the rectangles to a single point. In the case of the next figure, orbits might be double-counted, but the effect of that problem can be controlled, because the map is finite-to-one. Under the equivalence relation, this situation results in a branched manifold.



The inverse limit of the quotient space is topologically conjugate to the original space and also satisfies Axioms 1 and 2. What is interesting about this construction is that the quotient space is a branched manifold. Williams used a very similar construction to explain them, by placing an equivalence relation on a compact neighborhood in a foliated manifold.

5.2 Inverse Limits and Laminations

Rojo looked at Williams' result that an inverse limit of a branched manifold is homeomorphic to an expanding attractor of a diffeomorphism of a manifold ([LR13]). In [CRS11], Rojo, along with Cuesta and Stadler, obtained results by thinking of laminations as tiling spaces. He then adapted those results to codimension zero laminations and used tiling space results to prove his new results. As with Wieler, Rojo states a construction theorem and a realization theorem. Here, there is a stronger restriction on the initial space in the projective system. Before, the space was only required to be a compact metric space.

Theorem 5.2.1. *Fix a projective system* (B_k, f_k) *where* B_k *are branched* n*-manifolds and* f_k *cellular maps, both of class* C^r*. The inverse limit* B_∞ *of the system is a codimension zero lamination of dimension* n *and class* C^r *if and only if the systems is flattening.*

Theorem 5.2.2. *Any codimension zero lamination* $(M, \mathcal{L})$ *is homeomorphic to an inverse limit* $\varprojlim(S_k, f_k)$ *of branched manifolds* S_k *and cellular maps* $f_k : S_k \to S_{k-1}$*.*

Theorem 5.2.1 can be called a construction theorem because it constructs a certain type of lamination if we only have information about the projective system. Theorem 5.2.2 can be thought of as a realization theorem because it says there is a lamination homeomorphic to the projective system, but does not construct it.

The first theorem is very intuitive. Even though the initial spaces are branched manifolds, a lamination cannot have a branch point, so in order for the inverse limit to be a lamination, the branch point must be ironed out. Branch points must be mapped to branch points and the ‘flaps’ near the branch must eventually be mapped to each other.

In the proof of theorem 5.2.2, branched manifolds are constructed in a remarkably similar fashion to the proof of Theorem B. In Wieler’s proof, there existed a Markov partition on the space because it was irreducible. Here, there exists a simplicial box decomposition for the lamination because it has codimension zero. If needed, there are ways to modify the map at this step to become cellular. The transversals of the boxes are collapsed using an equivalence relation, just as before, and the result is a branched manifold. A sequence of branched manifolds is constructed inductively and the inverse limit of this projective system is homeomorphic to the lamination.

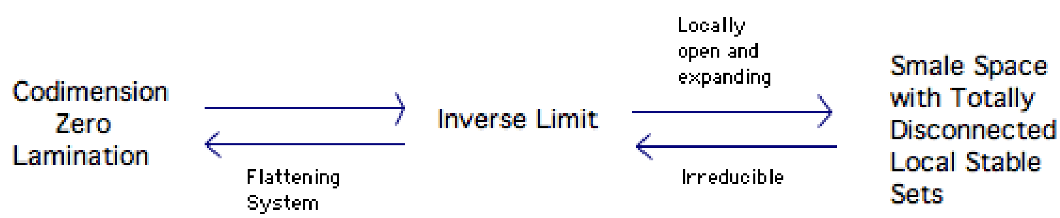
5.3 Connecting both results

Propositions 4.3.1 and 4.3.2 show that the totally disconnected and zero dimensional are just about equivalent. This relationship clues us in to how the stable and unstable sets of the Smale space relate to the lamination. In the lamination, there are two distinct directions: along a transversal and along a leaf. The transversal in the lamination is the stable set of the Smale space. Notice that this is the direction, in the proofs of both Theorem B and Theorem 5.2.2, in which the space was collapsed obtain a branched manifold. This is the contracting direction. Then it is clear that the codimension of the lamination is equal to the dimension of the stable set of the Smale space.. Additionally, the leaf in the lamination is the same direction as the unstable set, which is the ”expanding” direction.

The flattening condition can also be explained in terms of both laminations and

Smale spaces. Theorem 5.2.1 insists that the projective system be flattening, in order for the inverse limit to be a codimension zero lamination. This strongly relates to Proposition 3.3.5. that says $X^u(x,\epsilon) \times X^s(x,\epsilon)$ is homeomorphic to a neighborhood of x. So the branch point must be flattened so that its neighborhood is homeomorphic to a Euclidean one.

The figure below summarizes the relationship between laminations, inverse limits and Smale spaces.



From left to right, any codimension zero lamination is homeomorphic to an inverse limit of a projective system with branched manifolds and cellular maps. If the projective system is locally open and expanding, then the inverse limit is a Smale space with totally disconnected local stable sets.

From right to left, a Smale space with totally disconnected local stable sets is topologically conjugate to an inverse limit if it is irreducible, and the projective system will satisfy Axiom 1 and 2. If the projective system is flattening, then the inverse limit is a codimension zero lamination.

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